

Tensor Analysis Practice questions -4

1. Suppose that $\{v_i\}$ and $\{\tilde{v}_i\}$ are ordered bases for a (finite dimensional real) vector space V , and that $T : V \rightarrow V$ is a linear transformation. Define scalars

$$\{A_j^i, T_j^i, \tilde{T}_j^i \in \mathbf{R} \mid 1 \leq i, j \leq \dim V\}$$

by the equations $v_j = A_j^i \tilde{v}_i$, $T(v_j) = T_j^i v_i$ and $T(\tilde{v}_j) = \tilde{T}_j^i \tilde{v}_i$. Find the transformation rule relating A_j^i , T_j^i , and \tilde{T}_j^i .

2. (Rossmann P. 21#1c) Suppose that $\{v_i\}$ and $\{\tilde{v}_i\}$ are ordered bases for a vector space V , and $\{f^i\}$ and $\{\tilde{f}^i\}$ their ordered dual bases for V^* respectively. Suppose $v_i = A_i^j \tilde{v}_j$ for some scalars $\{A_i^j \in \mathbf{R} \mid 1 \leq i, j \leq \dim V\}$. Find the transformation rule for the coordinates of covectors $f \in V^*$.

3. Rossmann P. 22#3

4. Suppose we are given 3 vector spaces V , W , and another we denote $V \otimes W$, together with a bilinear map $\otimes : V \times W \rightarrow V \otimes W$. As usual, denote $\otimes(v, w) = v \otimes w$ for $(v, w) \in V \times W$, and consider the following three conditions. Make no assumptions on dimension here.

- \otimes_1 For any vector space X and any bilinear map $\varphi : V \times W \rightarrow X$, there exists a unique linear map $f : V \otimes W \rightarrow X$ such that $f \otimes = \varphi$
- \otimes_2 For any vector space X and any bilinear map $\varphi : V \times W \rightarrow X$, there exists a linear map $f : V \otimes W \rightarrow X$ such that $f \otimes = \varphi$
- \otimes_3 $\text{span}\{v \otimes w \mid v \in V, w \in W\} = V \otimes W$.

- a) Prove $\otimes_1 \Rightarrow (\otimes_2 \text{ and } \otimes_3)$.
- b) Prove that $(\otimes_2 \text{ and } \otimes_3) \Rightarrow \otimes_1$.

5. Consider the bilinear map $\otimes : \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{M}_{m,n}(\mathbf{R})$ defined by

$$\otimes(y, x) = yx^t,$$

where x and y are column vectors (i.e. $n \times 1$ and $m \times 1$ matrices resp.), x^t denotes the transpose and yx^t denotes the matrix product. Prove that $(\mathbf{M}_{m,n}(\mathbf{R}), \otimes)$ is a tensor product of \mathbf{R}^n and \mathbf{R}^m .

6. Let S be any set and define $\langle S \rangle = \{f : S \rightarrow \mathbf{R} \mid f(s) \neq 0 \text{ for only finitely many } s \in S\}$. We know that $\langle S \rangle$ is a vector space. Prove that if we identify $f \in \langle S \rangle$ with the formal (finite) sum $\sum_{s \in S} f(s)s$, then the subset $S \subset \langle S \rangle$ is a basis for $\langle S \rangle$.

7. Suppose that $\{v_i\}$ and $\{w_i\}$ are bases for (finite dimensional) vector spaces V and W respectively, and let $\tilde{S} = \{v_i \otimes w_j \mid 1 \leq i \leq \dim V, 1 \leq j \leq \dim W\}$ be the finite set of symbols obtained from these bases as in class. For $v = x^i v_i$ and $w = y^j w_j$, define $\otimes : V \times W \rightarrow \langle \tilde{S} \rangle =: V \otimes W$, by

$$\otimes(v, w) = v \otimes w = x^i y^j v_i \otimes w_j.$$

- Prove that \otimes is bilinear.
- Prove that \tilde{S} is a basis for $\langle \tilde{S} \rangle$.
- Prove that $\langle \tilde{S} \rangle$ is a tensor product of V and W .
- Suppose that $\{v'_i\}$ and $\{w'_i\}$ are bases for V and W respectively. Show that $S' = \{v'_i \otimes w'_j \in V \otimes W \mid 1 \leq i \leq \dim V, 1 \leq j \leq \dim W\}$ is also a basis for $\langle \tilde{S} \rangle$.

8. Suppose $t = \sum_{i=1}^m u_i \otimes z_i \in V \otimes W$ and that $\{u_i\}$ is linearly independent. Show that $t = 0 \iff z_i = 0$ for all i .

9. a) Show that $0 \otimes w = v \otimes 0 = 0$ for all $v \in V, w \in W$.
 b) If $v \otimes w \neq 0$, show that $v \otimes w = v' \otimes w'$ iff $v' = \lambda v$ and $w' = \lambda^{-1} w$ for some $0 \neq \lambda \in \mathbf{R}$.

10. Let V be a vector space, and $v_1, \dots, v_p, v'_1, \dots, v'_p \in V$.

- Show that $v_1 \otimes v_2 \otimes \dots \otimes v_p = 0$ iff $v_i = 0$ for some i .
- Show that if $0 \neq v_1 \otimes v_2 \otimes \dots \otimes v_p$, then $v_1 \otimes v_2 \otimes \dots \otimes v_p = v'_1 \otimes v'_2 \otimes \dots \otimes v'_p$ iff $v'_i = \lambda_i v_i$ for some scalars λ_i satisfying $\lambda_1 \lambda_2 \dots \lambda_p = 1$.

11. Let V be a vector space, and let $V^{\otimes p} = V \otimes \dots \otimes V$ (p -times). Recall the definition from class of the subspace (actually a double-sided ideal) of $T(V)$ (that we quotient out by – “set to zero” – to obtain ΛV):

$$N = \text{span}\{a \otimes (u \otimes v + v \otimes u) \otimes b \mid v, w \in V \text{ and } a \in V^{\otimes p}, b \in V^{\otimes q}; p, q \in \mathbf{N}\}$$

Now define

$$M = \text{span}\{a \otimes w \otimes b \otimes w \otimes c \mid w \in V, a \in V^{\otimes p}, b \in V^{\otimes q}, c \in V^{\otimes r}; p, q, r \in \mathbf{N}\}$$

In the following, $a \in V^{\otimes p}, b \in V^{\otimes q}$ and $c \in V^{\otimes r}$.

- By setting $w = u + v$, show that $N \subset M$.
- Show that if $w \in V$, then

$$a \otimes w \otimes w \otimes b \otimes c - (-1)^q a \otimes w \otimes b \otimes w \otimes c \in N.$$

- Put $u = v$ to show that $\forall w \in V, a \otimes w \otimes w \otimes b \in N$.

- d) Use (b),(c) to show that $\forall w \in V, a \otimes w \otimes b \otimes w \otimes c \in N$
 e) Conclude from (d) that $M \subset N$.
 f) Conclude from (a) and (e) that $N = M$.

12. Recall that the *rank* of a tensor $t \in V \otimes W$ is the least m such that $t = \sum_{i=1}^m v_i \otimes w_i$ for vectors $v_i \in V$ and $w_i \in W$.

- a) Show that for any $0 \neq t \in V \otimes W$, we may write $t = \sum_{i=1}^m v_i \otimes w_i$ where $\{v_i \mid i = 1, \dots, m\}$ is linearly independent.
 b) Now show that for any $0 \neq t \in V \otimes W$, we may write $t = \sum_{i=1}^m v_i \otimes w_i$ where *both* $\{v_i \mid i = 1, \dots, m\}$ and $\{w_i \mid i = 1, \dots, m\}$ are linearly independent.
 c) Now prove that $\text{rank } t \leq \min\{\dim V, \dim W\}, \forall t \in V \otimes W$

13. Suppose V and W are finite dimensional. Recall the isomorphism $W \otimes V^* \xrightarrow{e} \text{Hom}(V, W)$ satisfying $e(w \otimes f)(v) = f(v)w$. The rank of a tensor is defined in problem 12, and recall that the rank of a linear transformation is the dimension of its image.

Prove that $\text{rank } t = \text{rank } e(t)$ for all $t \in W \otimes V^*$.

14. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

- a) Now, (using Q. 5) $A \in \mathbf{M}_{35} = \mathbf{R}^3 \otimes \mathbf{R}^5$, so write $A = \sum_{i=1}^m v_i \otimes w_i$ for some vectors $v_i \in \mathbf{R}^3, w_i \in \mathbf{R}^5$, with $m > 2$.
 b) Now write $A = \sum_{i=1}^m v_i \otimes w_i$ for some vectors $v_i \in \mathbf{R}^3, w_i \in \mathbf{R}^5$, with $m = 2$.

(Hint 1. Use the same technique that worked in Q.12 to reduce the “length” of the expression for A , i.e. do some work to make both $\{v_i\}$ and $\{w_i\}$ linearly independent.

Hint 2. The rank of A is 2, so we can write $A = P^{-1}\tilde{A}$, where $\tilde{A} = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}$ is in row echelon form, and $P = [c_1 \ c_2 \ c_3]$ is an invertible 3 by 3 matrix with the c_i as its columns. Recall/note that P is obtained from the identity by applying the same row operations that took A to \tilde{A} .)

15. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ and define $T \in \text{Hom}(\mathbf{R}^3, \mathbf{R}^3)$ by $T(v) = Av$.

Let $\mathbf{R}^3 \otimes \mathbf{R}^3 = \text{span}\{e_i \otimes e_j \mid 1 \leq i, j \leq 3\}$ be the usual tensor product, and recall that $f : \mathbf{R}^3 \otimes \mathbf{R}^3 \rightarrow \mathbf{M}_{33}(\mathbf{R})$ defined by $f(v \otimes w) = vw^t$ is an isomorphism.

Recall also that the unique linear map $e : \mathbf{R}^3 \otimes (\mathbf{R}^3)^* \rightarrow \text{Hom}(\mathbf{R}^3, \mathbf{R}^3)$ satisfying $e(e_i \otimes e^j)(v) = e^j(v)e_i$ is also an isomorphism. Let $t = e^{-1}(T)$.

- a) Find an explicit expression for $f^{-1}(A) \in \mathbf{R}^3 \otimes \mathbf{R}^3$.
- b) Find an explicit expression for $t \in \mathbf{R}^3 \otimes (\mathbf{R}^3)^*$.
- c) Write $t = \sum_{i=1}^m v_i \otimes w^i$ for $v_i \in \mathbf{R}^3, w^i \in (\mathbf{R}^3)^*$, where $m = \text{rank}(t)$.
- d) If $v \otimes w^*$ and $u \otimes x^*$ are two tensors in $\mathbf{R}^3 \otimes (\mathbf{R}^3)^*$, find an explicit expression for $e^{-1}(e(v \otimes w^*) \circ (u \otimes x^*))$, where \circ denotes the composition of the linear maps in $\text{Hom}(\mathbf{R}^3, \mathbf{R}^3)$.

16. In the following, $\{e_1, \dots, e_n\}$ and $\{e^1, \dots, e^n\}$ will denote the standard dual bases of \mathbf{R}^n and $(\mathbf{R}^n)^*$.

- a) Show that $\text{rank}(e_1 \otimes e_2 + e_2 \otimes e_1) = 2$.
- b) Find a tensor of rank 3 in $\mathbf{R}^3 \otimes \mathbf{R}^3$. You must show that the rank of your choice is 3.
- c) Find a tensor of rank 3 in $\mathbf{R}^3 \otimes (\mathbf{R}^4)^*$. You must show that the rank of your choice is 3.

17. Using the map $e : W \otimes V^* \rightarrow \text{Hom}(V, W)$, find the composition rule

$$(W \otimes V^*) \times (U \otimes W^*) \rightarrow U \otimes V^*$$

which corresponds to composition of linear maps.

18. Show that if $D_2 : \mathbf{R}^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}$ is defined by $D_2(v, w) = \det[v \ w]$ (where v, w are written as columns), then when viewed (using the maps j and e from class) as an element in $(\mathbf{R}^2)^* \otimes (\mathbf{R}^2)^*$,

$$D_2 = e^1 \otimes e^2 - e^2 \otimes e^1,$$

where $\{e_1, e_2\}$ is the standard basis of \mathbf{R}^2 , and $\{e^1, e^2\}$ its dual basis.

19. If $D_3 : \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}$ is defined by $D_3(u, v, w) = \det[uvw]$ (where u, v, w are written as columns), find an expression in $(\mathbf{R}^3)^* \otimes (\mathbf{R}^3)^* \otimes (\mathbf{R}^3)^*$ representing D_3 .

20. Suppose $T : V \rightarrow V$ is a linear map. If $\dim V = n$ show that for $n = 2, 3$ the induced map $\Lambda^n T : \Lambda^n V \rightarrow \Lambda^n V$ is multiplication by $\det T$.

- 21.** a) Find a form of rank 2 in $\Lambda^2 \mathbf{R}^4$. You must show that the rank of your choice is 2.
- b) Show that every form in $\Lambda^2 \mathbf{R}^3$ has rank 1.

22. Let $\{v_i\}$ be a basis for a finite dimensional vector space V and $\{v^i\}$ be its dual basis. Suppose g is an inner product on V . Using the natural isomorphism $(V \otimes V)^* \cong V^* \otimes V^*$, write $g = g_{ij} v^i \otimes v^j$. We know there is an isomorphism $\psi_g : V \rightarrow V^*$ induced by g . Show that $\psi_g(v_i) = g_{ij} v^j$ for $i = 1, \dots, \dim V$.

23. Let $\{v_1, \dots, v_n\}$ be an ordered orthonormal basis of the inner product space V , and suppose that $\star : \Lambda V \rightarrow \Lambda V$ is the Hodge star map associated to the given ordered basis. Show that

- a) $\star\star : \Lambda^p V \rightarrow \Lambda^p V$ is multiplication by ± 1 , and find the exact dependence of the sign on p and n .
- b) $\langle \alpha, \beta \rangle = \langle \alpha \wedge \star\beta \rangle$ for all $\alpha, \beta \in \Lambda^p V$.
- c) $\langle \star\alpha, \star\beta \rangle = \langle \alpha, \beta \rangle$, for all $\alpha, \beta \in \Lambda^p V$.

24. Let $\varphi : V \times V \rightarrow \mathbf{R}$ be a bilinear map.

- a) Show that $\tilde{\beta} : V^{2p} \rightarrow \mathbf{R}$ defined by

$$\tilde{\beta}(v_1, v_2, \dots, v_p, w_1, w_2, \dots, w_p) = \det[\varphi(v_i, w_j)],$$

is a multilinear map.

- b) Show that there is a unique bilinear map $\bar{\beta} : V^{\otimes p} \times V^{\otimes p} \rightarrow \mathbf{R}$ satisfying

$$\bar{\beta}(v_1 \otimes v_2 \otimes \dots \otimes v_p, w_1 \otimes w_2 \otimes \dots \otimes w_p) = \det[\varphi(v_i, w_j)].$$

- c) Show that if N is defined as in Q.11, then $\bar{\beta}(s, n) = \bar{\beta}(n, s) = 0$ for all $s \in V^{\otimes p}$ and $n \in N^p$.
- d) Show that there is a unique bilinear map $\beta : \Lambda^p V \times \Lambda^p V \rightarrow \mathbf{R}$ satisfying

$$\beta(v_1 \wedge v_2 \wedge \dots \wedge v_p, w_1 \wedge w_2 \wedge \dots \wedge w_p) = \det[\varphi(v_i, w_j)].$$

25. Let V be a vector space of dimension n .

- a) Show that $\{u_1, \dots, u_k, v_1, \dots, v_k\}$ is linearly independent iff $a = \sum_{i=1}^k u_i \wedge v_i$ satisfies $a^k \neq 0$.
- b) For any set of vectors $\{u_1, \dots, u_k, v_1, \dots, v_k\} \subset V$, show that $a = \sum_{i=1}^k u_i \wedge v_i$ satisfies $a^{k+1} = 0$.
- c) Prove that for $a \in \Lambda^2 V$, $\text{rank } a = \max\{k \mid a^k \neq 0\}$,
- d) Prove that for $a \in \Lambda^2 V$, $\text{rank } a \leq \frac{n}{2}$.
- e) Prove that if $v_1 \wedge v_2 \wedge \dots \wedge v_k$ and $w_1 \wedge w_2 \wedge \dots \wedge w_k$ are non-zero rank-one elements of $\Lambda^k V$, then

$$\exists \lambda \neq 0 \text{ s.t. } v_1 v_2 \dots v_k = \lambda w_1 w_2 \dots w_k \iff \text{span}\{v_1, v_2, \dots, v_k\} = \text{span}\{w_1, w_2, \dots, w_k\}.$$

26. a) Suppose $X \xrightarrow{h} Y$ and $Z \xrightarrow{k} U$ are linear maps. Explain briefly why there is a well defined linear map $h \otimes k : X \otimes Z \rightarrow Y \otimes U$ satisfying

$$(h \otimes k)(x \otimes y) = h(x) \otimes k(y), \quad \forall x \in X, y \in Y.$$

Now let V and W be vector spaces, and $f : V \rightarrow W$ a linear map.

b) Prove that f induces a well-defined linear map $\hat{f} : T(V) \rightarrow T(W)$ satisfying

$$\hat{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n), \quad \forall v_i \in V,$$

which also makes the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow & & \downarrow \\ T(V) & \xrightarrow{\hat{f}} & T(W) \end{array}$$

(Here the vertical maps are the usual inclusions $U \hookrightarrow T(U)$, for any vector space U .)

c) If $W \xrightarrow{i_W} \Lambda W$ and $V \xrightarrow{i_V} \Lambda V$ denote these usual inclusions, show that f induces a well-defined linear map $\tilde{f} : \Lambda V \rightarrow \Lambda W$ satisfying

$$\tilde{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n), \quad \forall v_i \in V,$$

which also makes the following diagram commute:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ i_V \downarrow & & \downarrow i_W \\ \Lambda V & \xrightarrow{\tilde{f}} & \Lambda W \end{array}$$

d) Suppose X and Y are subspaces of V such that $V = X \oplus Y$, with inclusion maps $X \xrightarrow{i} V$ and $Y \xrightarrow{j} V$. Show that if $\mu : \Lambda V \otimes \Lambda V \rightarrow \Lambda V$ denotes the multiplication map in ΛV , i.e., $\mu(\alpha \otimes \beta) = \alpha \wedge \beta$, then the composition $\Psi = \mu \circ (\tilde{i} \otimes \tilde{j})$

$$\Lambda X \otimes \Lambda Y \xrightarrow{\tilde{i} \otimes \tilde{j}} \Lambda V \otimes \Lambda V \xrightarrow{\mu} \Lambda V = \Lambda(X \oplus Y)$$

is an isomorphism.

27. If $j : V^* \otimes W^* \rightarrow (V \otimes W)^*$ is the map (which is an isomorphism when $\dim V + \dim W < \infty$) defined in class, show that if $f : V \rightarrow U$ and $g : W \rightarrow X$ are linear maps, then the following diagram commutes:

$$\begin{array}{ccc} V^* \otimes W^* & \xleftarrow{f^* \otimes g^*} & U^* \otimes X^* \\ j \downarrow & & j \downarrow \\ (V \otimes W)^* & \xleftarrow{(f \otimes g)^*} & (U \otimes X)^* \end{array}$$

28. Let V be a vector space, v a non-zero vector in V , and $f \in V^*$ any linear form satisfying $f(v) = 1$. Define a linear map $D : \Lambda V \rightarrow \Lambda V$ by

$$D(\alpha) = v \wedge \alpha, \quad \forall \alpha \in \Lambda V.$$

- a) Show that $\text{im } D \subset \ker D$ (i.e., $D^2 = 0$.)
- b) Show that we may write $V = \text{span}\{v\} \oplus \ker f$.
- c) Denote $Y = \ker f$. Assuming the results of Q.27d, show that for any element $\alpha \in \Lambda V$, there are unique elements $\beta_0, \beta_1 \in \Lambda Y$ such that

$$\alpha = v \wedge \beta_0 + \beta_1.$$

- d) Show that $\text{im } D = \ker D$.

29. Rossmann's exercises 1.1: 12, 13. (Note that we the space of $n \times n$ matrices $M_{n,n}$ is denoted $\mathbf{R}^{n \times n}$ in those notes.

30. Rossmann's exercises 1.1: 15. The Jacobian matrix is the matrix of partial derivatives.

31. Rossmann's exercises 1.1: 18.

32. Rossmann's exercises 1.2: 8

33. Prove that \mathbf{S}^2 is a connected 2 dimensional manifold. Is there an atlas with just 2 charts? Is there an atlas with just 1 chart?

34. Rossmann's exercises 1.2: 8 (modified) Let M be the set \mathbf{R} with the usual topology. Give \mathbf{R} the atlas $\{(\text{id}_{\mathbf{R}}, \mathbf{R})\}$ and denote the corresponding manifold M_0 (this is the usual manifold ' \mathbf{R} ').

- a) Define $\varphi : M \rightarrow \mathbf{R}$ by $\varphi(t) = t^3$.
 - i) Show that $\{(\varphi, M)\}$ is an atlas for a manifold structure on M .
 - ii) Is $f = \text{id}_{\mathbf{R}} : M \rightarrow M_0$ a diffeomorphism?
 - iii) Viewing f as a map $f : M \rightarrow M$, is f in $C^\infty(M)$?
 - iv) Is $g = \text{id}_{\mathbf{R}} : M_0 \rightarrow M$ a C^∞ map?
 - v) Can you find a diffeomorphism $h : M_0 \rightarrow M$? If so, exhibit one.
- b) Define $\psi_1 : M \setminus \{0\} \rightarrow \mathbf{R}$ by $\psi_1(t) = t^3$ and $\psi_2 : (-1, 1) \rightarrow \mathbf{R}$ by $\psi_2(t) = \frac{t}{1-t}$.
 - i) Show that $\{(\psi_1, M \setminus \{0\}), (\psi_2, (-1, 1))\}$ is an atlas for a manifold structure on M .
 - ii) Is $f = \text{id}_{\mathbf{R}} : M \rightarrow M_0$ a diffeomorphism?
 - iii) Viewing f as a map $f : M \rightarrow M$, is f in $C^\infty(M)$?
 - iv) Is $g = \text{id}_{\mathbf{R}} : M_0 \rightarrow M$ a C^∞ map?
 - v) Can you find a diffeomorphism $k : M_0 \rightarrow M$? If so, exhibit one.

35-45. Text: problems 2-12, inclusive.

46. Suppose (U, φ) is a coordinate chart on an smooth n -Manifold M , and let $\psi : W = \varphi(U) \rightarrow U$ denote φ^{-1} . Suppose $v \in \text{Vect}(W)$.

Define $\psi_*(v) : C^\infty(U) \rightarrow C^\infty(U)$ by

$$\psi_*(v)(f) = [v(f \circ \psi)] \circ \varphi.$$

Show that $\psi_*(v) \in \text{Vect}(U)$, and that $\psi_* : \text{Vect}(W) \rightarrow \text{Vect}(U)$ so defined is an invertible linear map.

47. Let $M = \mathbf{R}^2$ with the standard differentiable structure (i.e. with the unique maximal atlas containing the chart $(\mathbf{R}^2, \text{id}_{\mathbf{R}^2})$.) If $v = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$, show that the flow $\phi_t : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ generated by v is $\phi_t \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, i.e. rotation about the origin through an angle t in the counterclockwise sense.

48. Suppose M is a smooth manifold, and let $v, w \in \text{Vect}(M)$. Define $vw : C^\infty(M) \rightarrow C^\infty(M)$ by

$$vw(f) = v(w(f))$$

- Show by an example with $M = \mathbf{R}$ that $vw \notin \text{Vect}(M)$
- Show that if we define $[v, w] = vw - wv$, for $v, w \in \text{Vect}(M)$, that $[v, w] \in \text{Vect}(M)$
- Given (2 different) examples for $M = \mathbf{R}$, one where $[v, w] = 0$, and another where $[v, w] \neq 0$.
- For arbitrary vector fields $v, w \in \text{Vect}(\mathbf{R})$, find necessary and sufficient conditions for $[v, w] = 0$.

49. For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{2,2}$, define $v^A \in \text{Vect}(\mathbf{R}^2)$ by

$$v^A = (ax + cy) \frac{\partial}{\partial x} + (bx + dy) \frac{\partial}{\partial y}.$$

Prove that if $A, B \in \mathbf{M}_{2,2}$, then the Lie bracket $[v^A, v^B] = v^{[A, B]}$, where $[A, B] = AB - BA$ is the usual commutator of A and B in $\mathbf{M}_{2,2}$.

50-67. Text: problems 13, 14, 16, 17, 18 (assume ϕ is a diffeomorphism), 19 (but see the errata for the book, as there's a mistake), 20, 21, 22, 23 (check the errata again), 24, 25, 27, 28, 29, 33, 34.

(*) For questions 68-74 (if necessary): Suppose (U, φ) is a coordinate chart on an smooth n -Manifold M , and let $\psi : \varphi(U) \rightarrow U$ denote φ^{-1} . Define, as in class, $\frac{\partial}{\partial z^i} = \psi_* \left(\frac{\partial}{\partial x^i} \right)$, and $z^i \in C^\infty(U)$ by $z^i = x^i \circ \varphi$.

68. a) Show that $\frac{\partial z^i}{\partial z^j} = \delta_j^i$.

b) If dz^i denotes the differential of the local coordinate function z^i , show that $dz^i(\frac{\partial}{\partial z^i}) = \delta_j^i$.

69. Show that if $v \in \text{Vect}(U)$, then $v = v^k \frac{\partial}{\partial z^k}$ for some smooth functions $v^k \in C^\infty(U)$.

70. Let (U, φ) be a local coordinate system for a smooth manifold M . If $p \in U$, let $\epsilon = \sup \{r \mid B(\varphi(p), r) \subset \varphi(U)\}$. If $\varphi(p) = (a_1, \dots, a_n)$, define

$$\beta_i(t) = \psi(a_1, \dots, a_{i-1}, a_i + t, a_{i+1}, \dots, a_n)$$

for $t \in (-\epsilon, \epsilon)$.

Show that $\beta_i'(0) = \frac{\partial}{\partial z^i}(p)$, as members of $T_p M$.

71. a) Suppose (U, φ) is a coordinate chart on an smooth n -manifold M , and let $\psi : W = \varphi(U) \rightarrow U$ denote φ^{-1} . Suppose $v \in \text{Vect}(W)$.

Define $\psi_*(v) : C^\infty(U) \rightarrow C^\infty(U)$ by

$$\psi_*(v)(f) = [v(f \circ \psi)] \circ \varphi.$$

We know that $\psi_* : \text{Vect}(W) \rightarrow \text{Vect}(U)$ is an invertible linear map.

Define $\frac{\partial}{\partial z^i} = \psi_*\left(\frac{\partial}{\partial x^i}\right)$, as usual.

Show carefully that $\frac{\partial^2 f}{\partial z^i \partial z^j} := \frac{\partial}{\partial z^i}\left(\frac{\partial}{\partial z^j}(f)\right)$ for $f \in C^\infty(U)$ satisfies

$$\frac{\partial^2 f}{\partial z^i \partial z^j} = \frac{\partial^2 f}{\partial z^j \partial z^i}, \quad \text{for all } 1 \leq i, j \leq n, \text{ and } f \in C^\infty(U).$$

b) Now suppose $\psi : M \rightarrow N$ is a diffeomorphism with inverse φ , and define $\psi_* : \text{Vect}(M) \rightarrow \text{Vect}(N)$ by the formula in (a). Show that $\psi_*([v, w]) = [\psi_*v, \psi_*w]$ for all vector fields $v, w \in \text{Vect}(M)$. (i.e. ψ_* preserves the Lie bracket of vector fields.)

72. (a) If $i : \mathbf{S}^2 \rightarrow \mathbf{R}^3$ is the inclusion map, show that it is a smooth map. (\mathbf{S}^2 and \mathbf{R}^3 have the usual manifold structures)

b) Show that $i_* : T_p \mathbf{S}^2 \rightarrow T_p \mathbf{R}^3$ is an injective linear map. (This allows us to identify $T_p \mathbf{S}^2$ with the subspace $i_*(T_p \mathbf{S}^2)$ of $T_p \mathbf{R}^3$.)

73. For $M = \mathbf{S}^1$, let $U = \{(x, y) \in \mathbf{S}^1 \mid x > 0\}$ and $\theta : U \rightarrow \mathbf{R}$ be defined by $\theta(x, y) = \arctan(\frac{y}{x})$. (Here, $\arctan : \mathbf{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.)

- i) Show that θ is a homeomorphism onto $(-\frac{\pi}{2}, \frac{\pi}{2})$, by finding $\psi \equiv \theta^{-1}$.
- ii) Define $\frac{\partial}{\partial \theta} \in \text{Vect}(U)$ by $\frac{\partial}{\partial \theta} = \psi_*\left(\frac{d}{dt}\right)$, where $t : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbf{R}$ is the usual coordinate $t(x) = x$. (If you like, $t \equiv x^1$.)

Show carefully that if $a = (x_0, y_0) \in U$, and $i : U \rightarrow \mathbf{R}^2$ is the (smooth) inclusion, then

$$i_*\left(\frac{\partial}{\partial \theta_a}\right) := -y_0 \frac{\partial}{\partial x_a} + x_0 \frac{\partial}{\partial y_a}.$$

74. Let $M = \mathbf{S}^2$, $U = \{(x, y, z) \in \mathbf{S}^2 \mid x > 0\}$ and define $\varphi : U \rightarrow \varphi(U)$ by $\varphi(x, y, z) = (y, z) = (x^1, x^2)$, with inverse ψ . As usual, define $\frac{\partial}{\partial z^k} = \psi_*\left(\frac{\partial}{\partial x^k}\right)$. If $j : \mathbf{S}^2 \rightarrow \mathbf{R}^3$ is the smooth inclusion, Compute $(j_*\frac{\partial}{\partial z^k})_p$ for $k = 1, 2$ in terms of the tangent vectors $\frac{\partial}{\partial x_p}, \frac{\partial}{\partial y_p}, \frac{\partial}{\partial z_p}$ on \mathbf{R}^3 , using the identification we employed in class (and justified by exercise 71.)

75. We know (Warner, P. 10, and assignment 2) that for any $r > 0$, there is $k \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\forall v, \|v\| \leq r \Rightarrow k(v) = 1$, and $\forall v, \|v\| > 2r \Rightarrow k(v) = 0$. Let M be a smooth manifold.

- a) Show that $\forall p \in M$, and any open set $U \ni p$ there is an open set $V \ni p$ with $V \subset \tilde{V} \subset U$, and a smooth function $f \in \mathbf{C}^\infty(M)$ such that $f(p) = 1$ on V , and $f = 0$ in $M \setminus U$.
- b) Prove that if $g \in \mathbf{C}^\infty(M)$ is zero on an open set containing p , then $\forall v_p \in T_p M, v_p(g) = 0$. (Hint: Show that there is a function $h \in \mathbf{C}^\infty(M)$ with $h(p) = 1$ and $0 = hg$.)
- c) Suppose (φ, U) is a coordinate system for M , and $p \in U$. Prove that for every $f \in \mathbf{C}^\infty(U)$, there is $\tilde{f} \in \mathbf{C}^\infty(M)$ and an open set $W \ni p$ such that $f = \tilde{f}$ on W .
- d) Suppose (φ, U) is a coordinate system for M , and $p \in U$. Prove that $i_* : T_p U \rightarrow T_p M$ is an isomorphism, where $i : U \hookrightarrow M$ denotes the inclusion.

76. Let $\omega = \omega_i dx^i \in \Omega^1(\mathbf{R}^n)$ be a 1-form on \mathbf{R}^n , and $\beta : [a, b] \rightarrow \mathbf{R}^n$ a smooth curve (since $[a, b]$ isn't open in \mathbf{R} , this means the usual thing.) We define $\int_\beta \omega$, the *integral of ω over β* as

$$\int_\beta \omega = \int_a^b \omega(\beta'(t)) dt$$

Note that the expression on the right hand side is the ordinary (Riemann) integral of the smooth function $\omega(\beta'(t))$, and $\omega(\beta'(t))$ is simply the value of the one-form ω on the

tangent vector $\beta'(t) \in T_{\beta(t)} \mathbf{R}^n$. If $\beta(t) = \beta^j(t)e_j$, so that $\beta'(t) = \frac{d\beta^j}{dt} \frac{\partial}{\partial x^j} \beta(t)$, an explicit expression for $\omega(\beta'(t))$ is given by

$$\omega(\beta'(t)) = \omega_i(\beta(t)) dx^i(\beta'(t)) = \omega_i(\beta(t)) dx^i \left(\frac{d\beta^j}{dt} \frac{\partial}{\partial x^j} \beta(t) \right) = \omega_i(\beta(t)) \frac{d\beta^i}{dt}.$$

Hence,

$$\int_{\beta} \omega = \int_a^b \omega_i(\beta(t)) \frac{d\beta^i}{dt} dt.$$

For $n = 2, 3$ you'll recognize this from your second year calculus course as a *line integral*, but there the object being integrated was a vector field with the same components as ω . (We didn't tell you then, but you were really integrating 1-forms. Some authors are courageous enough to call expressions like $F_1 dx + F_2 dy + F_3 dz$ differential forms -e.g. Marsden, Tromba and Weinstein who wrote a good second year text of which you may still have a copy. You'll also recall that the integral $\int_{\beta} \omega$ is independent of the parametrization of the curve $\beta(t)$. See Marsden, Tromba and Weinstein, 'Basic Multivariable Calculus' P.361-362)

Now let $M = \mathbf{R}^2 \setminus \{0\}$, and define a curve β in M by $\beta(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ for $t \in [0, 2\pi]$.

a) Define

$$\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \in \Omega^1(M).$$

Show that $\int_{\beta} \omega = 2\pi$.

b) Show that $dw = 0$

c) Show that if $\omega \in \Omega^1(M)$ and $\omega = d\theta$ for some smooth function $\theta \in C^\infty(M)$, then $\int_{\beta} \omega = 0$. Conclude that $\omega \neq d\theta$ for any smooth function $\theta \in C^\infty(M)$.

d) Now let $N = \mathbf{R}^2 \setminus \{(x, y) \mid x \leq 0\}$, and define $\sigma = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \in \Omega^1(N)$. Show that there is a smooth function $\theta \in C^\infty(N)$ such that $\sigma = d\theta$.

77. Let V and W be finite-dimensional vector spaces, and $T : V \rightarrow W$ an injective linear map.

a) Show that $T^* : W^* \rightarrow V^*$ is surjective.

b) Show that $\Lambda^k T^* : \Lambda^k W^* \rightarrow \Lambda^k V^*$ is surjective for every $k \geq 0$.

78. Define $\beta \in \Omega^1(\mathbf{R}^3)$ by

$$\beta = x dx + y dy + z dz$$

Now let $i : \mathbf{S}^2 \rightarrow \mathbf{R}^3$ denote the inclusion map. (We shall identify p and $i(p)$ in the following when convenient.)

- b) Show that, for all $p \in \mathbf{S}^2$, the map $i_p^* : T_p^* \mathbf{R}^3 \rightarrow T_p^* \mathbf{S}^2$ is onto, but is not injective. Find $\dim \ker i_p^*$ without using part (c).
- c) Show that $i^*(\beta) = 0$, and hence that $\ker i_p^* = \text{span}\{\beta_p\}$.
- d) Use (c) to show then $i_p^*(dx)_p = 0$ at $p = (1, 0, 0)$. To avoid all the subscripts, This is usually written as $i^*(dx) = 0$ at $p = (1, 0, 0)$. Indeed, find all $p \in \mathbf{S}^2$ where $i^*(dx) = di^*(x) = 0$. (The distinction between $di^*(x)$ and dx is rarely made. One usually says, for example, “ dx restricted to \mathbf{S}^2 is zero at $p = (1, 0, 0)$ ”)

79. Let $M = \mathbf{S}^2 \setminus \{(0, y, z) \in \mathbf{S}^2 \mid y \leq 0\}$ and $W = (0, \pi) \times (-\pi, \pi) \subset \mathbf{R}^2$. A diffeomorphism $\psi : W \rightarrow M$ is defined by

$$\psi(u, v) = (\cos v \sin u, \sin v \sin u, \cos u).$$

Let $\phi : M \rightarrow W$ denote its inverse, and define coordinate functions $\varphi, \theta : M \rightarrow \mathbf{R}$ as usual by $\phi(p) = (\varphi(p), \theta(p))$, for $p \in M$.

We identify $T_p M$ with the subspace $i_*(T_p M)$ of $T_p \mathbf{R}^3$ using the inclusion $i : M \rightarrow \mathbf{R}^3$ as usual.

- a) Show carefully that

$$\begin{aligned} \frac{\partial}{\partial \varphi} &:= \psi_* \left(\frac{\partial}{\partial u} \right) = \cos \theta \cos \varphi \frac{\partial}{\partial x} + \sin \theta \cos \varphi \frac{\partial}{\partial y} - \sin \varphi \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &:= \psi_* \left(\frac{\partial}{\partial v} \right) = -\sin \theta \sin \varphi \frac{\partial}{\partial x} + \cos \theta \sin \varphi \frac{\partial}{\partial y} \end{aligned}$$

- b) Use the fact that $(x, y, z) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$, (or any other valid argument) to show that

$$\begin{aligned} d\varphi &= i^* \left(-\frac{1}{\sin \varphi} dz \right) \quad \text{and} \\ d\theta &= i^* \left(\frac{1}{\sin \varphi} (-\sin \theta dx + \cos \theta dy) \right) \end{aligned}$$

80. Let $\omega = \omega_i dx^i \in \Omega^1(\mathbf{R}^n)$ be a 1-form on \mathbf{R}^n , and $\beta : [a, b] \rightarrow \mathbf{R}^n$ a smooth curve (since $[a, b]$ isn't open in \mathbf{R} , this means the usual thing.) We define $\int_\beta \omega$, the *integral of ω over β* as

$$\int_\beta \omega = \int_a^b \omega(\beta'(t)) dt$$

Note that the expression on the right hand side is the ordinary (Riemann) integral of the smooth function $\omega(\beta'(t))$, and $\omega(\beta'(t))$ is simply the value of the one-form ω on the tangent vector $\beta'(t) \in T_{\beta(t)}\mathbf{R}^n$. If $\beta(t) = \beta^j(t)e_j$, so that $\beta'(t) = \frac{d\beta^j}{dt} \frac{\partial}{\partial x^j} \beta(t)$, an explicit expression for $\omega(\beta'(t))$ is given by

$$\omega(\beta'(t)) = \omega_i(\beta(t)) dx^i(\beta'(t)) = \omega_i(\beta(t)) dx^i \left(\frac{d\beta^j}{dt} \frac{\partial}{\partial x^j} \beta(t) \right) = \omega_i(\beta(t)) \frac{d\beta^i}{dt}.$$

Hence,

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For $n = 2, 3$ you'll recognize this from your second year calculus course as a *line integral*, but there the object being integrated was a vector field with the same components as ω . (We didn't tell you then, but you were really integrating 1-forms. Some authors are courageous enough to call expressions like $F_1 dx + F_2 dy + F_3 dz$ differential forms -e.g. Marsden, Tromba and Weinstein who wrote a good second year text of which you may still have a copy. You'll also recall that the integral $\int_{\beta} \omega$ is independent of the parametrization of the curve $\beta(t)$. See Marsden, Tromba and Weinstein, 'Basic Multivariable Calculus' P.361-362)

Now let $M = \mathbf{R}^2 \setminus \{0\}$, and define a curve β in M by $\beta(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ for $t \in [0, 2\pi]$.

a) Define

$$\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \in \Omega^1(M).$$

Show that $\int_{\beta} \omega = 2\pi$.

b) Show that $dw = 0$

c) Show that if $\omega \in \Omega^1(M)$ and $\omega = d\theta$ for some smooth function $\theta \in C^\infty(M)$, then $\int_{\beta} \omega = 0$. Conclude that $\omega \neq d\theta$ for any smooth function $\theta \in C^\infty(M)$.

d) Now let $N = \mathbf{R}^2 \setminus \{(x, y) \mid x \leq 0\}$, and define $\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \in \Omega^1(N)$. Show that there is a smooth function $\theta \in C^\infty(N)$ such that $\omega = d\theta$.

81. Let V and W be finite-dimensional vector spaces, and $T : V \rightarrow W$ an injective linear map.

a) Show that $T^* : W^* \rightarrow V^*$ is surjective.

b) Show that $\Lambda^k T^* : \Lambda^k W^* \rightarrow \Lambda^k V^*$ is surjective for every $k \geq 0$.

82. Define $\beta \in \Omega^1(\mathbf{R}^3)$ by

$$\beta = x dx + y dy + z dz$$

Now let $i : \mathbf{S}^2 \rightarrow \mathbf{R}^3$ denote the inclusion map. (We shall identify p and $i(p)$ in the following when convenient.)

b) Show that, for all $p \in \mathbf{S}^2$, the map $i_p^* : T_p^* \mathbf{R}^3 \rightarrow T_p^* \mathbf{S}^2$ is onto, but is not injective. Find $\dim \ker i_p^*$ without using part (c).

c) Show that $i^*(\beta) = 0$, and hence that $\ker i_p^* = \text{span}\{\beta_p\}$.

d) Use (c) to show then $i_p^*(dx)_p = 0$ at $p = (1, 0, 0)$. To avoid all the subscripts, This is usually written as $i^*(dx) = 0$ at $p = (1, 0, 0)$. Indeed, find all $p \in \mathbf{S}^2$ where $i^*(dx) = di^*(x) = 0$. (The distinction between $di^*(x)$ and dx is rarely made. One usually says, for example, “ dx restricted to \mathbf{S}^2 is zero at $p = (1, 0, 0)$ ”)

83. Let $M = \mathbf{S}^2 \setminus \{(0, y, z) \in \mathbf{S}^2 \mid y \leq 0\}$ and $W = (0, \pi) \times (-\pi, \pi) \subset \mathbf{R}^2$. A diffeomorphism $\psi : W \rightarrow M$ is defined by

$$\psi(u, v) = (\cos v \sin u, \sin v \sin u, \cos u).$$

Let $\phi : M \rightarrow W$ denote its inverse, and define coordinate functions $\varphi, \theta : M \rightarrow \mathbf{R}$ as usual by $\phi(p) = (\varphi(p), \theta(p))$, for $p \in M$.

We identify $T_p M$ with the subspace $i_*(T_p M)$ of $T_p \mathbf{R}^3$ using the inclusion $i : M \rightarrow \mathbf{R}^3$ as usual.

a) Show carefully that

$$\begin{aligned} \frac{\partial}{\partial \varphi} &:= \psi_* \left(\frac{\partial}{\partial u} \right) = \cos \theta \cos \varphi \frac{\partial}{\partial x} + \sin \theta \cos \varphi \frac{\partial}{\partial y} - \sin \varphi \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &:= \psi_* \left(\frac{\partial}{\partial v} \right) = -\sin \theta \sin \varphi \frac{\partial}{\partial x} + \cos \theta \sin \varphi \frac{\partial}{\partial y} \end{aligned}$$

b) Use the fact that $(x, y, z) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$, (or any other valid argument) to show that

$$\begin{aligned} d\varphi &= -\frac{1}{\sin \varphi} dz \quad \text{and} \\ d\theta &= \frac{1}{\sin \varphi} (-\sin \theta dx + \cos \theta dy) \end{aligned}$$

77-88. Text exercises 48, 50, 51, 58, 59, 63, 64, 65, 66, 67, 70, 71

89. Consider the smooth map $\phi : (0, 1) \times (-\pi, \pi) \times (0, \pi) \rightarrow \{v \in \mathbf{R}^3 \mid \|v\| \leq 1\}$ defined by

$$\phi(r, \varphi, \theta) = r(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$$

Show that $\phi^*(dx \wedge dy \wedge dz) = r^2 \sin \varphi dr \wedge d\varphi \wedge d\theta$.

90. Define forms on \mathbf{R}^3 by

$$\begin{aligned}\beta &= x dx + y dy + z dz \\ \alpha &= x dy \wedge dz + z dx \wedge dy + y dz \wedge dx.\end{aligned}$$

- a) Let $M = \mathbf{S}^2 \setminus \{(0, y, z) \in \mathbf{S}^2 \mid y \leq 0\}$ and let $\theta, \varphi \in C^\infty(M)$ be the functions satisfying $(x, y, z) = (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$ for $(x, y, z) \in M$. Consider the (smooth) inclusion map $j : M \rightarrow \mathbf{R}^3$. Show that $j^*(\alpha) = \sin \varphi d\varphi \wedge d\theta$.
- b) Use Q. 74 to show that

$$i^*(x dx \wedge dy - z dy \wedge dz) = i^*(x dx \wedge dz + y dy \wedge dz) = i^*(y dx \wedge dy + z dx \wedge dz) = 0.$$

- f) Use (b) (and 74(b)) to conclude that $i_p^* : \Lambda^2 T_p^* \mathbf{R}^3 \rightarrow \Lambda^2 T_p^* \mathbf{S}^2$ is non-zero for all $p \in \mathbf{S}^2$.
- g) Show that $i^*(\alpha)$ is a nowhere-vanishing 2-form on \mathbf{S}^2 , i.e. $i_p^*(\alpha_p) \neq 0$ for all $p \in \mathbf{S}^2$.

91. Let $M = \mathbf{S}^2 \setminus \{(0, y, z) \in \mathbf{S}^2 \mid y \leq 0\}$.

- a) Identify $T_p \mathbf{R}^3$ with \mathbf{R}^3 in the usual way: $\frac{\partial}{\partial x^i} \mapsto e_i$, where $\{e_1, e_2, e_3\}$ is the standard ordered basis of \mathbf{R}^3 . Now define $g(u, v) = i_*(v) \cdot i_*(v)$ for $v, w \in T_p M$ (where “ \cdot ” denotes the standard inner product on \mathbf{R}^3). Show that $g = d\varphi \otimes d\varphi + \sin^2 \varphi d\theta \otimes d\theta$ and that this defines a Riemannian metric on M .
- b) Show that $\omega = \sin \varphi d\varphi \wedge d\theta$ is a volume form on M (so that M is oriented), and that it is the canonical volume form on M .
- c) Find $\star(d\varphi)$ and $\star(d\theta)$, and hence compute $\star(dx)$, $\star(dy)$ and $\star(dz)$.