Tensor Analysis   Practice questions -3

1. Suppose that $\{v_i\}$ and $\{\tilde{v}_i\}$ are ordered bases for a (finite dimensional real) vector space V, and that $T : V \to V$ is a linear transformation. Define scalars

$$\{A^i_j, T^i_j, \tilde{T}^i_j \in \mathbf{R} \mid 1 \leq i, j \leq \dim V\}$$

by the equations $v_j = A^i_j \tilde{v}_i$, $T(v_j) = T^i_j v_i$ and $T(\tilde{v}_j) = \tilde{T}^i_j \tilde{v}_i$. Find the transformation rule relating A^i_j, T^i_j, and $\tilde{T}^i_j$.

2. (Rossmann P. 21#1c) Suppose that $\{v_i\}$ and $\{\tilde{v}_i\}$ are ordered bases for a vector space V, and $\{f^i\}$ and $\{\tilde{f^i}\}$ their ordered dual bases for V^* respectively. Suppose $v_i = A^j_i \tilde{v}_j$ for some scalars $\{A^j_i \in \mathbf{R} \mid 1 \leq i, j \leq \dim V\}$. Find the transformation rule for the coordinates of covectors $f \in V^*$.

3. Rossmann P. 22#3

4. Suppose we are given 3 vector spaces V, W, and another we denote $V \otimes W$, together with a a bilinear map $\otimes : V \times W \to V \otimes W$. As usual, denote $\otimes(v, w) = v \otimes w$ for $(v, w) \in V \times W$, and consider the following three conditions. Make no assumptions on dimension here.

- $\otimes_1$ For any vector space X and any bilinear map $\varphi : V \times W \to X$, there exists a unique linear map $f : V \otimes W \to X$ such that $f\otimes = \varphi$
- $\otimes_2$ For any vector space X and any bilinear map $\varphi : V \times W \to X$, there exists a linear map $f : V \otimes W \to X$ such that $f\otimes = \varphi$
- $\otimes_3$ $\operatorname{span}\{v \otimes w \mid v \in V, w \in W\} = V \otimes W$.

a) Prove $\otimes_1 \Rightarrow (\otimes_2 \text{ and } \otimes_3)$.
b) Prove that $(\otimes_2 \text{ and } \otimes_3) \Rightarrow \otimes_1$.

5. Consider the bilinear map $\otimes : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{M}_{m,\,n}(\mathbf{R})$ defined by

$$\otimes(y, x) = yx^t,$$

where x and y are column vectors (i.e. $n \times 1$ and $m \times 1$ matrices resp.), x^t denotes the transpose and yx^t denotes the matrix product. Prove that $(\mathbf{M}_{m,\,n}(\mathbf{R}), \otimes)$ is a tensor product of $\mathbf{R}^n$ and $\mathbf{R}^m$.

6. Let S be any set and define $\langle S \rangle = \{f : S \to \mathbf{R} \mid f(s) \neq 0 \text{ for only finitely many } s \in S\}$. We know that $\langle S \rangle$ is a vector space. Prove that if we identify $f \in \langle S \rangle$ with the formal (finite) sum $\sum_{s \in S} f(s)s$, then the subset $S \subset \langle S \rangle$ is a basis for $\langle S \rangle$.

7. Suppose that $\{v_i\}$ and $\{w_i\}$ are bases for (finite dimensional) vector spaces V and W respectively, and let $\tilde{S} = \{v_i \otimes w_j \mid 1 \leq i \leq \dim V, 1 \leq j \leq \dim W\}$ be the finite set of symbols obtained from these bases as in class. For $v = x^i v_i$ and $w = y^j w_j$, define $\otimes : V \times W \to \langle \tilde{S} \rangle =: V \otimes W$, by

$$\otimes(v, w) = v \otimes w = x^i y^j v_i \otimes w_j.$$

a) Prove that $\otimes$ is bilinear.
b) Prove that $\tilde{S}$ is a basis for $\langle \tilde{S} \rangle$.
c) Prove that $\langle \tilde{S} \rangle$ is a is a tensor product of V and W.
c) Suppose that $\{v'_i\}$ and $\{w'_i\}$ are bases for V and W respectively. Show that $S' = \{v'_i \otimes w'_j \in V \otimes W \mid 1 \leq i \leq \dim V, 1 \leq j \leq \dim W\}$ is also a basis for $\langle \tilde{S} \rangle$.

8. Suppose $t = \sum_{i=1}^m u_i \otimes z_i \in V \otimes W$ and that $\{u_i\}$ is linearly independent. Show that $t = 0 \iff z_i = 0$ for all i.

9. a) Show that $0 \otimes w = v \otimes 0 = 0$ for all $v \in V, w \in W$.
b) If $v \otimes w \neq 0$, show that $v \otimes w = v' \otimes w'$ iff $v' = \lambda v$ and $w' = \lambda^{-1} w$ for some $0 \neq \lambda \in \mathbf{R}$.

10. Let V be a vector space, and $v_1, \ldots v_p, v'_1, \ldots v'_p \in V$.

a) Show that $v_1 \otimes v_2 \otimes \cdots \otimes v_p = 0$ iff $v_i = 0$ for some i.
b) Show that if $0 \neq v_1 \otimes v_2 \otimes \cdots \otimes v_p$, then $v_1 \otimes v_2 \otimes \cdots \otimes v_p = v'_1 \otimes v'_2 \otimes \cdots \otimes v'_p$ iff $v'_i = \lambda_i v_i$ for some scalars λ_i satisfying $\lambda_1 \lambda_2 \ldots \lambda_p = 1$.

11. Let V be a vector space, and let $V^{\otimes^p} = V \otimes \cdots \otimes V$ (p-times). Recall the definition from class of the subspace (actually a double-sided ideal) of $T(V)$ (that we quotient out by – "set to zero" – to obtain ΛV):

$$N = \mathrm{span}\{a \otimes (u \otimes v + v \otimes u) \otimes b \mid v, w \in V \text{ and } a \in V^{\otimes^p}, b \in V^{\otimes^q}; p, q \in \mathbf{N}\}$$

Now define

$$M = \mathrm{span}\{a \otimes w \otimes b \otimes w \otimes c \mid w \in V, a \in V^{\otimes^p}, b \in V^{\otimes^q}, c \in V^{\otimes^r}; p, q, r \in \mathbf{N}\}$$

In the following, $a \in V^{\otimes^p}, b \in V^{\otimes^q}$ and $c \in V^{\otimes^r}$.

a) By setting $w = u + v$, show that $N \subset M$.
b) Show that if $w \in V$, then

$$a \otimes w \otimes w \otimes b \otimes c - (-1)^q a \otimes w \otimes b \otimes w \otimes c \in N.$$

c) Put $u = v$ to show that $\forall w \in V, \quad a \otimes w \otimes w \otimes b \in N.$

d) Use (b),(c) to show that $\forall w \in V, a \otimes w \otimes b \otimes w \otimes c \in N$
e) Conclude from (d) that $M \subset N$.
f) Conclude from (a) and (e) that $N = M$.

12. Recall that the *rank* of a tensor $t \in V \otimes W$ is the least m such that $t = \sum_{i=1}^m v_i \otimes w_i$ for vectors $v_i \in V$ and $w_i \in W$.

a) Show that for any $0 \neq t \in V \otimes W$, we may write $t = \sum_{i=1}^m v_i \otimes w_i$ where $\{v_i \mid i = 1, \ldots m\}$ is linearly independent.
b) Now show that for any $0 \neq t \in V \otimes W$, we may write $t = \sum_{i=1}^m v_i \otimes w_i$ where *both* $\{v_i \mid i = 1, \ldots m\}$ and $\{w_i \mid i = 1, \ldots m\}$ are linearly independent.
c) Now prove that $\operatorname{rank} t \leq \min\{\dim V, \dim W\}, \forall t \in V \otimes W$

13. Suppose V and W are finite dimensional. Recall the isomorphism $W \otimes V^* \xrightarrow{e} \operatorname{Hom}(V, W)$ satisfying $e(w \otimes f)(v) = f(v)w$. The rank of a tensor is defined in problem 12, and recall that the rank of a linear transformation is the dimension of its image.

Prove that $\operatorname{rank} t = \operatorname{rank} e(t)$ for all $t \in W \otimes V^*$.

14. Let

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

a) Now, (using Q. 5) $A \in \mathbf{M}_{3\,5} = \mathbf{R}^3 \otimes \mathbf{R}^5$, so write $A = \sum_{i=1}^m v_i \otimes w_i$ for some vectors $v_i \in \mathbf{R}^3, w_i \in \mathbf{R}^5$, with $m > 2$.

b) Now write $A = \sum_{i=1}^m v_i \otimes w_i$ for some vectors $v_i \in \mathbf{R}^3, w_i \in \mathbf{R}^5$, with $m = 2$.

(Hint 1. Use the same technique that worked in Q.12 to reduce the "length" of the expression for A, i.e. do some work to make both $\{v_i\}$ and $\{w_i\}$ linearly independent.

Hint 2. The rank of A is 2, so we can write $A = P^{-1}\tilde{A}$, where $\tilde{A} = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}$ is in row echelon form, and $P = [c_1\, c_2\, c_3]$ is an invertible 3 by 3 matrix with the c_i as it columns. Recall/note that P is obtained from the identity by applying the same row operations that took A to $\tilde{A}$.)

15. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ and define $T \in \operatorname{Hom}(\mathbf{R}^3, \mathbf{R}^3)$ by $T(v) = Av$.

Let $\mathbf{R}^3 \otimes \mathbf{R}^3 = \operatorname{span}\{e_i \otimes e_j \mid 1 \leq i, j \leq 3\}$ be the usual tensor product, and recall that $f : \mathbf{R}^3 \otimes \mathbf{R}^3 \to \mathbf{M}_{33}(\mathbf{R})$ defined by $f(v \otimes w) = vw^t$ is an isomorphism.

Recall also that the unique linear map $e : \mathbf{R}^3 \otimes (\mathbf{R}^3)^* \to \operatorname{Hom}(\mathbf{R}^3, \mathbf{R}^3)$ satisfying $e(e_i \otimes e^j)(v) = e^j(v)e_i$ is also an isomorphism. Let $t = e^{-1}(T)$.

a) Find an explicit expression for $f^{-1}(A) \in \mathbf{R}^3 \otimes \mathbf{R}^3$.
b) Find an explicit expression for $t \in \mathbf{R}^3 \otimes (\mathbf{R}^3)^*$.
c) Write $t = \sum_{i=1}^{m} v_i \otimes w^i$ for $v_i \in \mathbf{R}^3, w^i \in (\mathbf{R}^3)^*$, where $m = \operatorname{rank}(t)$.
d) If $v \otimes w^*$ and $u \otimes x^*$ are two tensors in $\mathbf{R}^3 \otimes (\mathbf{R}^3)^*$, find an explicit expression for $e^{-1}(e(v \otimes w^*) \circ (u \otimes x^*))$, where $\circ$ denotes the composition of the linear maps in $\operatorname{Hom}(\mathbf{R}^3, \mathbf{R}^3)$.

16. In the following, $\{e_1, \ldots, e_n\}$ and $\{e^1, \ldots, e^n\}$ will denote the standard dual bases of $\mathbf{R}^n$ and $(\mathbf{R}^n)^*$.
a) Show that $\operatorname{rank}(e_1 \otimes e_2 + e_2 \otimes e_1) = 2$.
b) Find a tensor of rank 3 in $\mathbf{R}^3 \otimes \mathbf{R}^3$. You must show that the rank of your choice is 3.
c) Find a tensor of rank 3 in $\mathbf{R}^3 \otimes (\mathbf{R}^4)^*$. You must show that the rank of your choice is 3.

17. Using the map $e : W \otimes V^* \to \operatorname{Hom}(V, W)$, find the composition rule

$$(W \otimes V^*) \times (U \otimes W^*) \to U \otimes V^*$$

which corresponds to composition of linear maps.

18. Show that if $D_2 : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ is defined by $D_2(v, w) = \det[v\, w]$ (where v, w are written as columns), then when viewed (using the maps j and e from class) as an element in $(\mathbf{R}^2)^* \otimes (\mathbf{R}^2)^*$,

$$D_2 = e^1 \otimes e^2 - e^2 \otimes e^1,$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbf{R}^2$, and $\{e^1, e^2\}$ its dual basis.

19. If $D_3 : \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ is defined by $D_3(u, v, w) = \det[u\, v\, w]$ (where u, v, w are written as columns), find an expression in $(\mathbf{R}^3)^* \otimes (\mathbf{R}^3)^* \otimes (\mathbf{R}^3)^*$ representing D_3.

20. Suppose $T : V \to V$ is a linear map. If $\dim V = n$ show that for $n = 2, 3$ the induced map $\Lambda^n T : \Lambda^n V \to \Lambda^n V$ is multiplication by $\det T$.

21. a) Find a form of rank 2 in $\Lambda^2 \mathbf{R}^4$. You must show that the rank of your choice is 2.
b) Show that every form in $\Lambda^2 \mathbf{R}^3$ has rank 1.

22. Let $\{v_i\}$ be a basis for a finite dimensional vector space V and $\{v^i\}$ be its dual basis. Suppose g is an inner product on V. Using the natural isomorphism $(V \otimes V)^* \cong V^* \otimes V^*$, write $g = g_{i\,j} v^i \otimes v^j$. We know there is an isomorphism $\psi_g : V \to V^*$ induced by g. Show that $\psi_g(v_i) = g_{i\,j} v^j$ for $i = 1, \ldots, \dim V$.

23. Let $\{v_1, \ldots, v_n\}$ be an ordered orthonormal basis of the inner product space V, and suppose that $\star : \Lambda V \to \Lambda V$ is the Hodge star map associated to the given ordered basis. Show that

a) $\star\star : \Lambda^p V \to \Lambda^p V$ is multiplication by ± 1, and find the exact dependence of the sign on p and n.
b) $\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta)$ for all $\alpha, \beta \in \Lambda^p V$.
c) $\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle$, for all $\alpha, \beta \in \Lambda^p V$.

24. Let $\varphi : V \times V \to \mathbf{R}$ be a bilinear map.

a) Show that $\tilde{\beta} : V^{2p} \to \mathbf{R}$ defined by

$$\tilde{\beta}(v_1, v_2, \cdots, v_p, w_1, w_2, \cdots, w_p) = \det[\varphi(v_i, w_j)],$$

is a multilinear map.

b) Show that there is a unique bilinear map $\bar{\beta} : V^{\otimes^p} \times V^{\otimes^p} \to \mathbf{R}$ satisfying

$$\bar{\beta}(v_1 \otimes v_2 \otimes \cdots \otimes v_p, w_1 \otimes w_2 \otimes \cdots \otimes w_p) = \det[\varphi(v_i, w_j)].$$

c) Show that if N is defined as in Q.11, then $\bar{\beta}(s, n) = \bar{\beta}(n, s) = 0$ for all $s \in V^{\otimes^p}$ and $n \in N^p$.

d) Show that there is a unique bilinear map $\beta : \Lambda^p V \times \Lambda^p V \to \mathbf{R}$ satisfying

$$\beta(v_1 \wedge v_2 \wedge \cdots \wedge v_p, w_1 \wedge w_2 \wedge \cdots \wedge w_p) = \det[\varphi(v_i, w_j)].$$

25. Let V be a vector space of dimension n.

a) Show that $\{u_1, \ldots, u_k, v_1, \ldots v_k\}$ is linearly independent iff $a = \sum_{i=1}^k u_i \wedge v_i$ satisfies $a^k \neq 0$.

b) For any set of vectors $\{u_1, \ldots, u_k, v_1, \ldots v_k\} \subset V$, show that $a = \sum_{i=1}^k u_i \wedge v_i$ satisfies $a^{k+1} = 0$.

c) Prove that for $a \in \Lambda^2 V$, $\operatorname{rank} a = \max\{k \mid a^k \neq 0\}$,
d) Prove that for $a \in \Lambda^2 V, \operatorname{rank} a \leq \frac{n}{2}$.
e) Prove that if $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ and $w_1 \wedge w_2 \wedge \ldots \wedge w_k$ are non-zero rank–one elements of $\Lambda^k V$, then

$$\exists \lambda \neq 0 \text{ s.t. } v_1 v_2 \ldots v_k = \lambda w_1 w_2 \ldots w_k \iff \operatorname{span}\{v_1, v_2, \ldots, v_k\} = \operatorname{span}\{w_1, w_2, \ldots, w_k\}.$$

26. a) Suppose $X \xrightarrow{h} Y$ and $Z \xrightarrow{k} U$ are linear maps. Explain briefly why there is a well defined linear map $h \otimes k : X \otimes Z \to Y \otimes U$ satisfying

$$(h \otimes k)(x \otimes y) = h(x) \otimes k(y), \quad \forall x \in X, y \in Y.$$

Now let V and W be vector spaces, and $f : V \to W$ a linear map.

b) Prove that f induces a well-defined linear map $\hat{f} : T(V) \to T(W)$ satisfying

$$\hat{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n), \quad \forall v_i \in V,$$

which also makes the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] & W \arrow[d] \\ T(V) \arrow[r, "\hat{f}"] & T(W) \end{tikzcd}$$

(Here the vertical maps are the usual inclusions $U \hookrightarrow T(U)$, for any vector space U.)

c) If $W \xrightarrow{i_U} \Lambda W$ and $V \xrightarrow{i_U} \Lambda V$ denote these usual inclusions, show that f induces a well-defined linear map $\tilde{f} : \Lambda V \to \Lambda W$ satisfying

$$\tilde{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n), \quad \forall v_i \in V,$$

which also makes the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i_V"'] & W \arrow[d, "i_W"] \\ \Lambda V \arrow[r, "\tilde{f}"] & \Lambda W \end{tikzcd}$$

d) Suppose X and Y are subspaces of V such that $V = X \oplus Y$, with inclusion maps $X \xrightarrow{i} V$ and $Y \xrightarrow{j} V$. Show that if $\mu : \Lambda V \otimes \Lambda V \to \Lambda V$ denotes the multiplication mapin ΛV, i.e., $\mu(\alpha \otimes \beta) = \alpha \wedge \beta$, then the composition $\Psi = \mu \circ (\tilde{i} \otimes \tilde{j})$

$$\Lambda X \otimes \Lambda Y \xrightarrow{\tilde{i} \otimes \tilde{j}} \Lambda V \otimes \Lambda V \xrightarrow{\mu} \Lambda V = \Lambda(X \oplus Y)$$

is an isomorphism.

27. If $j : V^* \otimes W^* \to (V \otimes W)^*$ is the map (which is an isomorphism when $\dim V + \dim W < \infty$) defined in class, show that if $f : V \to U$ and $g : W \to X$ are linear maps, then the following diagram commutes:

$$\begin{tikzcd} V^* \otimes W^* \arrow[d, "j"'] & U^* \otimes X^* \arrow[l, "f^* \otimes g^*"'] \arrow[d, "j"'] \\ (V \otimes W)^* & (U \otimes X)^* \arrow[l, "(f \otimes g)^*"'] \end{tikzcd}$$

28. Let V be a vector space, v a non-zero vector in V, and $f \in V^*$ any linear form satisfying $f(v) = 1$. Define a linear map $D : \Lambda V \to \Lambda V$ by

$$D(\alpha) = v \wedge \alpha, \quad \forall \alpha \in \Lambda V.$$

a) Show that $\operatorname{im} D \subset \ker D$ (i.e., $D^2 = 0$.)
b) Show that we may write $V = \operatorname{span}\{v\} \oplus \ker f$.
c) Denote $Y = \ker f$. Assuming the results of Q.27d, show that for any element $\alpha \in \Lambda V$, there are unique elements $\beta_0, \beta_1 \in \Lambda Y$ such that

$$\alpha = v \wedge \beta_0 + \beta_1.$$

d) Show that $\operatorname{im} D = \ker D$.

29. Rossmann's exercises 1.1: 12, 13. (Note that we the space of $n \times n$ matrices $M_{n\,n}$ is denoted $\mathbf{R}^{n \times n}$ in those notes.

30. Rossmann's exercises 1.1: 15. The Jacobian matrix is the matrix of partial derivatives.

31. Rossmann's exercises 1.1: 18.

32. Rossmann's exercises 1.2: 8

33. Prove that $\mathbf{S}^2$ is a connected 2 dimensional manifold. Is there an atlas with just 2 charts? Is there an atlas with just 1 chart?

34. Rossmann's exercises 1.2: 8 (modified) Let M be the set $\mathbf{R}$ with the usual topology. Give $\mathbf{R}$ the atlas $\{(\mathrm{id}_R, \mathbf{R})\}$ and denote the corresponding manifold M_0 (this is the usual manifold '$\mathbf{R}$').

a) Define $\varphi : M \to \mathbf{R}$ by $\varphi(t) = t^3$.
   i) Show that $\{(\varphi, M)\}$ is an atlas for a manifold structure on M.
   ii) Is $f = \mathrm{id}_{\mathbf{R}} : M \to M_0$ a diffeomorphism?
   iii) Viewing f as a map $f : M \to M$, is f in $C^\infty(M)$?
   iv) Is $g = \mathrm{id}_{\mathbf{R}} : M_0 \to M$ a C^∞ map?
   v) Can you find a diffeomorphism $h : M_0 \to M$? If so, exhibit one.

b) Define $\psi_1 : M \setminus \{0\} \to \mathbf{R}$ by $\psi(t) = t^3$ and $\psi_2 : (-1, 1) \to \mathbf{R}$ by $\psi_2(t) = \frac{t}{1-t}$.
   i) Show that $\{(\psi_1, M \setminus \{0\}), (\psi_2, (-1, 1)\}$ is an atlas for a manifold structure on M.
   ii) Is $f = \mathrm{id}_{\mathbf{R}} : M \to M_0$ a diffeomorphism?
   iii) Viewing f as a map $f : M \to M$, is f in $C^\infty(M)$?
   iv) Is $g = \mathrm{id}_{\mathbf{R}} : M_0 \to M$ a C^∞ map?
   v) Can you find a diffeomorphism $k : M_0 \to M$? If so, exhibit one.

35-45. Text: problems 2-12, inclusive.

46. Suppose (U, φ) is a coordinate chart on an smooth n-Manifold M, and let $\psi : W = \varphi(U) \to U$ denote φ^{-1}. Suppose $v \in \mathrm{Vect}(W)$.

Define $\psi_*(v) : C^\infty(U) \to C^\infty(U)$ by

$$\psi_*(v)(f) = [v(f \circ \psi)] \circ \varphi.$$

Show that $\psi_*(v) \in \mathrm{Vect}(U)$, and that $\psi_* : \mathrm{Vect}(W) \to \mathrm{Vect}(U)$ so defined is an invertible linear map.

47. Let $M = \mathbf{R}^2$ with the standard differentiable structure (i.e. with the unique maximal atlas containing the chart $(\mathbf{R}^2, \mathrm{id}_{\mathbf{R}^2})$.) If $v = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$, show that the flow $\phi_t : \mathbf{R}^2 \to \mathbf{R}^2$ generated by v is $\phi_t(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, i.e. rotation about the origin through an angle t in the counterclockwise sense.

48. Suppose M is a smooth manifold, and let $v, w \in \mathrm{Vect}(M)$ Define $vw : C^\infty(M) \to C^\infty(M)$ by

$$vw(f) = v(w(f))$$

a) Show by an example with $M = \mathbf{R}$ that $vw \notin \mathrm{Vect}(M)$
b) Show that if we define $[v, w] = vw - wv$, for $v, w \in \mathrm{Vect}(M)$, that $[v, w] \in \mathrm{Vect}(M)$
c) Given (2 different) examples for $M = R$, one where $[v, w] = 0$, and another where $[v, w] \neq 0$.
d) For arbitrary vector fields $v, w \in \mathrm{Vect}(\mathbf{R})$, find necessary and sufficient conditions for $[v, w] = 0$.

49. For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{2\,2}$, define $v^A \in \mathrm{Vect}(\mathbf{R}^2)$ by

$$v^A = (ax + cy)\frac{\partial}{\partial x} + (bx + dy)\frac{\partial}{\partial y}.$$

Prove that if $A, B \in \mathbf{M}_{2\,2}$, then the Lie bracket $[v^A, v^B] = v^{[A,B]}$, where $[A, B] = AB - BA$ is the usual commutator of A and B in $\mathbf{M}_{2\,2}$.

50-67. Text: problems 13, 14, 16,17, 18 (assume ϕ is a diffeomorphism), 19 (but see the errata for the book, as there's a mistake), 20, 21, 22, 23 (check the errata again), 24, 25, 27, 28, 29, 33, 34.

($\star$) For questions 68-74 (if necessary): Suppose (U, φ) is a coordinate chart on an smooth n-Manifold M, and let $\psi : \varphi(U) \to U$ denote φ^{-1}. Define, as in class, $\dfrac{\partial}{\partial z^i} = \psi_*(\dfrac{\partial}{\partial x^i})$, and $z^i \in C^\infty(U)$ by $z^i = x^i \circ \varphi$.

68. a) Show that $\dfrac{\partial z^i}{\partial z^j} = \delta^i_j$.

b) If dz^i denotes the differential of the local coordinate function z^i, show that $dz^i(\dfrac{\partial}{\partial z^i}) = \delta^i_j$.

69. Show that if $v \in \mathrm{Vect}(U)$, then $v = v^k \dfrac{\partial}{\partial z^k}$ for some smooth functions $v^k \in C^\infty(U)$.

70. Let (U, φ) be a local coordinate system for a smooth manifold M. If $p \in U$, let $\epsilon = \sup\{r \mid B(\varphi(p), r) \subset \varphi(U)\}$. If $\varphi(p) = (a_1, \ldots, a_n)$, define

$$\beta_i(t) = \psi(a_1, \ldots, a_{i-1}, a_i + t, a_{i+1}, \ldots, a_n)$$

for $t \in (-\epsilon, \epsilon)$.

Show that $\beta_i'(0) = \dfrac{\partial}{\partial z^i}(p)$, as members of T_pM.

71. a) Suppose (U, φ) is a coordinate chart on an smooth n-manifold M, and let $\psi : W = \varphi(U) \to U$ denote φ^{-1}. Suppose $v \in \mathrm{Vect}(W)$.

Define $\psi_*(v) : C^\infty(U) \to C^\infty(U)$ by

$$\psi_*(v)(f) = [v(f \circ \psi)] \circ \varphi.$$

We know that $\psi_* : \mathrm{Vect}(W) \to \mathrm{Vect}(U)$ is an invertible linear map.

Define $\dfrac{\partial}{\partial z^i} = \psi_*(\dfrac{\partial}{\partial x^i})$, as usual.

Show carefully that $\dfrac{\partial^2 f}{\partial z^i\,\partial z^j} := \dfrac{\partial}{\partial z^i}(\dfrac{\partial}{\partial z^j}(f))$ for $f \in C^\infty(U)$ satisfies

$$\frac{\partial^2 f}{\partial z^i\,\partial z^j} = \frac{\partial^2 f}{\partial z^j\,\partial z^i}, \quad \text{for all } 1 \le i, j \le n, \text{ and } f \in C^\infty(U).$$

b) Now suppose $\psi : M \to N$ is a diffeomorphism with inverse φ, and define $\psi_* : \mathrm{Vect}(M) \to \mathrm{Vect}(N)$ by the formula in (a). Show that $\psi_*([v, w]) = [\psi_* v, \psi_* w]$ for all vector fields $v, w \in \mathrm{Vect}(M)$. (i.e. ψ_* preserves the Lie bracket of vector fields.)

72. (a) If $i : \mathbf{S}^2 \to \mathbf{R}^3$ is the inclusion map, show that is it a smooth map. ($\mathbf{S}^2$ and $\mathbf{R}^3$ have the usual manifold structures)

b) Show that $i_* : T_p\mathbf{S}^2 \to T_p\mathbf{R}^3$ is an injective linear map. (This allows us to identify $T_p\mathbf{S}^2$ with the subspace $i_*(T_p\mathbf{S}^2)$ of $T_p\mathbf{R}^3$.)

73. For $M = \mathbf{S}^1$, let $U = \{(x, y) \in \mathbf{S}^1 \mid x > 0\}$ and $\theta : U \to \mathbf{R}$ be defined by $\theta(x, y) = \arctan(\frac{y}{x})$. (Here, $\arctan : \mathbf{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.)

i) Show that θ is a homeomorphism onto $(-\frac{\pi}{2}, \frac{\pi}{2})$, by finding $\psi \equiv \theta^{-1}$.

ii) Define $\dfrac{\partial}{\partial \theta} \in \mathrm{Vect}(U)$ by $\dfrac{\partial}{\partial \theta} = \psi_*(\dfrac{d}{dt})$, where $t : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbf{R}$ is the usual coordinate $t(x) = x$. (If you like, $t \equiv x^1$.)

Show carefully that if $a = (x_0, y_0) \in U$, and $i : U \to \mathbf{R}^2$ is the (smooth) inclusion, then

$$i_*(\frac{\partial}{\partial \theta}\bigg|_a) := -y_0 \frac{\partial}{\partial x}\bigg|_a + x_0 \frac{\partial}{\partial y}\bigg|_a .$$

74. Let $M = \mathbf{S}^2$, $U = \{(x, y, z) \in \mathbf{S}^2 \mid x > 0\}$ and define $\varphi : U \to \varphi(U)$ by $\varphi(x, y, z) = (y, z) = (x^1, x^2)$, with inverse ψ. As usual, define $\dfrac{\partial}{\partial z^k} = \psi_*(\dfrac{\partial}{\partial x^k})$. If $j : \mathbf{S}^2 \to \mathbf{R}^3$ is the smooth inclusion, Compute $(j_* \dfrac{\partial}{\partial z^k})_p$ for $k = 1, 2$ in terms of the tangent vectors $\dfrac{\partial}{\partial x}\Big|_p, \dfrac{\partial}{\partial y}\Big|_p, \dfrac{\partial}{\partial z}\Big|_p$ on $\mathbf{R}^3$, using the identification we employed in class (and justified by exercise 71.)

75. We know (Warner, P. 10, and assignment 2) that for any $r > 0$, there is $k \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\forall v$, $\|v\| \le r \Rightarrow k(v) = 1$, and $\forall v$, $\|v\| > 2r \Rightarrow k(v) = 0$. Let M be a smooth manifold.

a) Show that $\forall p \in M$, and any open set $U \ni p$ there is an open set $V \ni p$ with $V \subset \bar{V} \subset U$, and a smooth function f $\in \mathbf{C}^\infty(M)$ such that $f(p) = 1$ on V, and $f = 0$ in $M \setminus U$.

b) Prove that if $g \in \mathbf{C}^\infty(M)$ is zero on an open set containing p, then $\forall v_p \in T_pM, v_p(g) = 0$. (Hint: Show that there is a function $h \in \mathbf{C}^\infty(M)$ with $h(p) = 1$ and $0 = hg$.)

c) Suppose (φ, U) is a coordinate system for M, and $p \in U$. Prove that for every $f \in \mathbf{C}^\infty(U)$, there is $\tilde{f} \in \mathbf{C}^\infty(M)$ and an open set $W \ni p$ such that $f = \tilde{f}$ on W.

d) Suppose (φ, U) is a coordinate system for M, and $p \in U$. Prove that $i_* : T_pU \to T_pM$ is an isomorphism, where $i : U \hookrightarrow M$ denotes the inclusion.

76. Let $\omega = \omega_i\, dx^i \in \Omega^1(\mathbf{R}^n)$ be a 1-form on $\mathbf{R}^n$, and $\beta : [a, b] \to \mathbf{R}^n$ a smooth curve (since $[a, b]$ isn't open in $\mathbf{R}$, this means the usual thing.) We define $\int_\beta \omega$, the *integral of* ω *over* β as

$$\int_\beta \omega = \int_a^b \omega(\beta'(t))dt$$

Note that the expression on the right hand side is the ordinary (Riemann) integral of the smooth function $\omega(\beta'(t))$, and $\omega(\beta'(t))$ is simply the value of the one-form ω on the tangent vector $\beta'(t) \in T_{\beta(t)}\mathbf{R}^n$. If $\beta(t) = \beta^j(t)e_j$, so that $\beta'(t) = \dfrac{d\beta^j}{dt}\dfrac{\partial}{\partial x^j}{}_{\beta(t)}$, an explicit expression for $\omega(\beta'(t))$ is given by

$$\omega(\beta'(t)) = \omega_i(\beta(t))\, dx^i(\beta'(t)) = \omega_i(\beta(t))\, dx^i\big(\frac{d\beta^j}{dt}\frac{\partial}{\partial x^j}{}_{\beta(t)}\big) = \omega_i(\beta(t))\frac{d\beta^i}{dt}.$$

Hence,

$$\int_\beta \omega = \int_a^b \omega_i(\beta(t))\frac{d\beta^i}{dt}dt.$$

For $n = 2, 3$ you'll recognize this from your second year calculus course as a *line integral*, but there the object being integrated was a vector field with the same components as ω. (We didn't tell you then, but you were really integrating 1-forms. Some authors are courageous enough to call expressions like $F_1dx + F_2dy + F_3dz$ differential forms -e.g. Marsden, Tromba and Weinstein who wrote a good second year text of which you may still have a copy. You'll also recall that the integral $\int_\beta \omega$ is independent of the parametrization of the curve $\beta(t)$. See Marsden, Tromba and Weinstein, 'Basic Multivariable Calculus' P.361-362)

Now let $M = \mathbf{R}^2 \setminus \{0\}$, and define a curve β in M by $\beta(t) = \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$ for $t \in [0, 2\pi]$.

a) Define

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(M).$$

Show that $\int_\beta \omega = 2\pi$.

c) Show that if $\omega \in \Omega^1(M)$ and $\omega = d\theta$ for some smooth function $\theta \in C^\infty(M)$, then $\int_\beta \omega = 0$. Conclude that $\omega \neq d\theta$ for any smooth function $\theta \in C^\infty(M)$.

d) Now let $N = \mathbf{R}^2 \setminus \{(x, y) \mid x \le 0\}$, and define $\sigma = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(N)$. Show that there is a smooth function $\theta \in C^\infty(N)$ such that $\sigma = d\theta$.

77. Let V and W be finite-dimensional vector spaces, and $T : V \to W$ an injective linear map.

a) Show that $T^* : W^* \to V^*$ is surjective.

b) Show that $\Lambda^k T^* : \Lambda^k W^* \to \Lambda^k V^*$ is surjective for every $k \ge 0$.

78. Define $\beta \in \Omega^1(\mathbf{R}^3)$ by

$$\beta = x\,dx + ydy + z\,dz$$

Now let $i : \mathbf{S}^2 \to \mathbf{R}^3$ denote the inclusion map. (We shall identify p and $i(p)$ in the following when convenient.)

b) Show that, for all $p \in \mathbf{S}^2$, the map $i_p^* : T_p^*\mathbf{R}^3 \to T_p^*\mathbf{S}^2$ is onto, but is not injective. Find $\dim \ker i_p^*$ without using part (c).

c) Show that $i^*(\beta) = 0$, and hence that $\ker i_p^* = \operatorname{span}\{\beta_p\}$.

d) Use (c) to show then $i_p^*(dx)_p = 0$ at $p = (1,0,0)$. To avoid all the subscripts, This is usually written as $i^*(dx) = 0$ at $p = (1,0,0)$. Indeed, find all $p \in \mathbf{S}^2$ where $i^*(dx) = di^*(x) = 0$. (The distinction between $di^*(x)$ and dx is rarely made. One usually says, for example, "dx restricted to $\mathbf{S}^2$ is zero at $p = (1,0,0)$")

79. Let $M = \mathbf{S}^2 \setminus \{(0,y,z) \in \mathbf{S}^2 \mid y \leq 0\}$ and $W = (0,\pi) \times (-\pi,\pi) \subset \mathbf{R}^2$. A diffeomorphism $\psi : W \to M$ is defined by

$$\psi(u,v) = (\cos v \sin u, \sin v \sin u, \cos u).$$

Let $\phi : M \to W$ denote its inverse, and define coordinate functions $\varphi, \theta : M \to \mathbf{R}$ as usual by $\phi(p) = (\varphi(p), \theta(p))$, for $p \in M$.

We identify T_pM with the subspace $i_*(T_pM)$ of $T_p\mathbf{R}^3$ using the inclusion $i : M \to \mathbf{R}^3$ as usual.

a) Show carefully that

$$\begin{aligned}
\frac{\partial}{\partial\varphi} &:= \psi_*(\frac{\partial}{\partial u}) = \cos\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\cos\varphi\frac{\partial}{\partial y} - \sin\varphi\frac{\partial}{\partial z} \\
\frac{\partial}{\partial\theta} &:= \psi_*(\frac{\partial}{\partial v}) = -\sin\theta\sin\varphi\frac{\partial}{\partial x} + \cos\theta\sin\varphi\frac{\partial}{\partial y}
\end{aligned}$$

b) Use the fact that $(x,y,z) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$, (or any other valid argument) to show that

$$\begin{aligned}
d\varphi &= -\frac{1}{\sin\varphi}\,dz \qquad \text{and} \\
d\theta &= \frac{1}{\sin\varphi}(-\sin\theta\,dx + \cos\theta\,dy)
\end{aligned}$$