Tensor Analysis Assignment 4: Due 4pm Friday 1-December.

1. Provide details for the parallel transport example on the Möbius strip presented in class on Wednesday 22-November. (This includes showing that the connection coefficents given in the two local charts give a well-defined connection on the intersection of the charts.)

In the questions 2 and 3, $M = \mathbf{S}^2 \setminus \{(x,0,z) \mid x \leq 0\}$ and $\theta, \varphi \in C^\infty(M)$ are the smooth functions satisfying $(x,y,z) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ for $(x,y,z) \in M$. Consider the inclusion map $i : M \to \mathbf{R}^3$, which is just the restriction of i to $M \subset \mathbf{S}^2$.

Let $G = dx \otimes dx + dy \otimes dy + dz \otimes dz$ be the standard metric on $\mathbf{R}^3$, and for all $p \in \mathbf{S}^2$ and $u, v \in T_pM$, define

$$g_p(u,v) = G_p(i_* u, i_* v).$$

Recall from assignment 3 that $g = d\varphi \otimes d\varphi + \sin^2\varphi \, d\theta \otimes d\theta$.

2. a) Show that the non-zero Christoffel symbols in the coordinates φ, θ for the Levi-Civita connection ∇ for g on M are

$$\Gamma^{\varphi}_{\theta\,\theta} = -\sin\varphi\cos\varphi, \quad \text{and } \Gamma^{\theta}_{\theta\,\varphi} = \Gamma^{\theta}_{\varphi\,\theta} = \cot\varphi$$

b) If $\dot{f}$ denotes $\dfrac{df}{dt}$, show that the geodesic equations are then

$$\ddot{\theta} + 2\dot{\varphi}\dot{\theta}\cot\varphi = 0 \quad \text{and} \quad \ddot{\varphi} - \dot{\theta}^2 \sin\varphi\cos\varphi = 0$$

c) Show that $\dot{\varphi}^2 + \dot{\theta}^2 \sin^2\varphi$ is constant along a geodesic. (Don't work hard!)

d) Show that the first equation in (b) can be written as $\dfrac{d(\dot{\theta}\sin^2\varphi)}{dt} = 0$.

e) Since $\dot{\theta}\sin^2\varphi$ is constant on a geodesic, show that when it is zero, the geodesic lies on a great circle which is a line of longitude.

3. (An example of parallel transport on $M \subset \mathbf{S}^2$.) Let $0 < \varphi_0 < \frac{\pi}{2}$, and let γ be the (piece-wise C^∞) curve on M defined in terms of the coordinates θ and ϕ by

$$(\varphi(\gamma(t)), \theta(\gamma(t))) = \begin{cases} ((t(\frac{\pi}{2} - \varphi_0) + \varphi_0), \frac{\pi}{2}), & 0 \le t \le 1 \\ (\frac{\pi}{2}, \frac{\pi}{2}(2-t)), & 1 \le t \le 2 \\ (t(\varphi_0 - \frac{\pi}{2}) - 2\varphi_0 + \frac{3\pi}{2}), 0), & 2 \le t \le 3 \end{cases}$$

Equip M with the Levi-Civita connection associated to the standard metric on $\mathbf{S}^2$ (Q.1). In the following, use (φ, θ) coordinates, and their Christoffel symbols computed in Q.2. Let $v_0 \in T_{\gamma(0)}M$ the vector field $\dfrac{\partial}{\partial \varphi}$ evaluated at $\gamma(0)$. .

a) Show that the parallel transport equations are

$$\frac{dv^\varphi}{dt} - \dot{\theta}\, v^\theta \sin\varphi \cos\varphi = 0 \quad \text{and}$$
$$\frac{dv^\theta}{dt} + \cot\varphi(\dot{\theta} v^\varphi + \dot{\varphi} v^\theta) = 0$$

b) Find $v_1 = P_\gamma(\gamma(0), \gamma(1))v_0$. (Hint: To solve for v^θ along this part, remember that a differential equation on the sort $\dfrac{dv^\theta}{dt} + g(t)v^\theta = 0$, with the initial condition $v^\theta(0) = 0$ has a unique solution. This unique solution is the obvious one. Or, if you wish to work hard, it can be solved using the 'integrating factor' $e^{\int f(t)dt}$.)

c) Find $v_2 = P_\gamma(\gamma(1), \gamma(2))v_1$

d) Find $v_3 = P_\gamma(\gamma(2), \gamma(3))v_2$. (Same hint as in (b))

e) Use Q. 76(a) to express v_0 and v_3 in terms of $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial z}$, remembering that they live in tangent spaces at different points.

f) Find $\lim_{\varphi_0 \to 0} v_0$ and $\lim_{\varphi_0 \to 0} v_3$