MAT 4183 Assignment 3: Due Wednesday 22-Nov-2017 at 6:15pm

A. Let M be a smooth manifold, TM its tangent bundle, and $v \in \mathrm{Vect}(M)$.

As usual, if $p \in M$, let v_p denote the tangent vector obtained from v at p. Prove that if we define $\tilde{v} : M \to TM$ by

$$\tilde{v}(p) = v_p, \quad \forall p \in M,$$

then $\tilde{v}$ is a *smooth section* of TM.

B:80^{++}. Let $M = \mathbf{R}^2 \setminus \{0\}$, $i : \mathbf{S}^1 \hookrightarrow M$ denote the (smooth) inclusion and define

$$\begin{aligned} \omega &= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(M), \quad \text{and} \\ \tilde{\omega} &= i^*\omega \in \Omega^1(\mathbf{S}^1) \end{aligned}$$

Define $\beta : \mathbf{R} \to M$ by $\beta(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ for $t \in \mathbf{R}$.

a) Show that $dw = 0$.

b) Show that $d\tilde{\omega} = 0$ *without* using the fact that $di^* = i^*d$.

c) Prove that $\tilde{\omega}$ is a nowhere vanishing 1-form on $\mathbf{S}^1$, i.e., that $\forall p \in \mathbf{S}^1, \tilde{\omega}_p \neq 0$.

d) If $s \in \mathbf{R}$, and $v_s \in T_s\mathbf{R}$ is $v_s = \frac{d}{dt}|_s$, show that $\beta_*(v_s) = \dot{\beta}(s) \in T_{\beta(s)}\mathbf{S}^1$.

e) Use (d) if necessary to prove that $\beta^*(\tilde{\omega}) = dt$.

f)* Suppose $\eta \in \Omega^1(\mathbf{S}^1)$ satisfies $\displaystyle\int_{\mathbf{S}^1} \eta = 0$. Show that $\eta = dh$ for some smooth function $h \in \mathbf{C}^\infty(\mathbf{S}^1)$. (You may assume that if $\tilde{h} \in \mathbf{C}^\infty(\mathbf{R})$ is periodic with period 2π, then $\tilde{h} = \beta^* h$ for some $h \in \mathbf{C}^\infty(\mathbf{S}^1)$.)

g)* Prove that the de-Rham cohomology of $\mathbf{S}^1$ is as was stated in lectures.

C:82. Define $\beta \in \Omega^1(\mathbf{R}^3)$ by

$$\beta = x\,dx + ydy + z\,dz$$

Now let $i : \mathbf{S}^2 \to \mathbf{R}^3$ denote the inclusion map. (We shall identify p and $i(p)$ in the following when convenient.)

b) Show that, for all $p \in \mathbf{S}^2$, the map $i_p^* : T_p^*\mathbf{R}^3 \to T_p^*\mathbf{S}^2$ is onto, but is not injective. Find $\dim \ker i_p^*$ without using part (c).

c) Show that $i^*(\beta) = 0$, and hence that $\ker i_p^* = \operatorname{span}\{\beta_p\}$.

d) Use (c) to show then $i_p^*(dx)_p = 0$ at $p = (1, 0, 0)$. To avoid all the subscripts, This is usually written as $i^*(dx) = 0$ at $p = (1, 0, 0)$. Indeed, find all $p \in \mathbf{S}^2$ where $i^*(dx) = di^*(x) = 0$. (The distinction between $di^*(x)$ and dx is rarely made. One usually says, for example, "dx restricted to $\mathbf{S}^2$ is zero at $p = (1, 0, 0)$")

D:89. Consider the smooth map $\phi : (0,1) \times (-\pi, \pi) \times (0, \pi) \to \{v \in \mathbf{R}^3 \mid ||v|| \leq 1\}$ defined by

$$\phi(r, \varphi, \theta) = r(\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

Show that $\phi^*(dx \wedge dy \wedge dz) = r^2 \sin\varphi \, dr \wedge d\varphi \wedge d\theta$.

E:90. Define forms on $\mathbf{R}^3$ by

$$\begin{aligned} \beta &= x\,dx + y dy + z\,dz \\ \alpha &= x\,dy \wedge dz + z\,dx \wedge dy + y\,dz \wedge dx. \end{aligned}$$

a) Let $M = \mathbf{S}^2 \setminus \{(0, y, z) \in \mathbf{S}^2 \mid y \leq 0\}$ and let $\theta, \varphi \in C^\infty(M)$ be the functions satisfying $(x, y, z) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$ for $(x, y, z) \in M$. Consider the (smooth) inclusion map $j : M \to \mathbf{R}^3$. Show that $j^*(\alpha) = \sin\varphi \, d\varphi \wedge d\theta$.

b) Show that

$$i^*(x\,dx \wedge dy - z\,dy \wedge dz) = i^*(x\,dx \wedge dz + y\,dy \wedge dz) = i^*(y\,dx \wedge dy + z\,dx \wedge dz) = 0.$$

c) Use 82(b) to conclude that $i_p^* : \Lambda^2 T_p^*\mathbf{R}^3 \to \Lambda^2 T_p^*\mathbf{S}^2$ is non-zero for all $p \in \mathbf{S}^2$.

d) Show that $i^*(\alpha)$ is a nowhere-vanishing 2-form on $\mathbf{S}^2$., i.e. $i_p^*(\alpha_p) \neq 0$ for all $p \in \mathbf{S}^2$.

F:91. Let $M = \mathbf{S}^2 \setminus \{(0, y, z) \in \mathbf{S}^2 \mid y \leq 0\}$.

a) Identify $T_p\mathbf{R}^3$ with $\mathbf{R}^3$ in the usual way: $\dfrac{\partial}{\partial x^i} \mapsto e_i$, where $\{e_1, e_2, e_3\}$ is the standard ordered basis of $\mathbf{R}^3$. Now define $g(u, v) = i_*(u) \cdot i_*(v)$ for $u, v \in T_pM$ (where "$\cdot$" denotes the standard inner product on $T\mathbf{R}^3 \cong \mathbf{R}^3 \times \mathbf{R}^3$). Show that $g = d\varphi \otimes d\varphi + \sin^2\varphi \, d\theta \otimes d\theta$ and that this defines a Riemannian metric on M.

b) Show that $\omega = \sin\varphi \, d\varphi \wedge d\theta$ is a volume form on M (so that M is oriented), and that it *is* the canonical volume form on M.

c) Find $\star(d\varphi)$ and $\star(d\theta)$, and hence compute $\star(dx), \star(dy)$ and $\star(dz)$.