MAT4183 Partial Solutions- Assignment 2

We assume throughout that all vector spaces are finite-dimensional.

24. Let $\{v_1, \ldots, v_n\}$ be an ordered basis of the vector space V, and suppose that

$$\star : \Lambda V \to \Lambda V$$

is the Hodge star map associated to this ordered basis.

Prove that

a) The composition $\star\star : \Lambda^p V \to \Lambda^p V$ is multiplication by ± 1, and find the exact dependence of the sign on p and n.

Solution: The key step involves the following computation, which yields $(-1)^{(n-p)p}$ as the appropriate sign.

$$\begin{aligned}
\operatorname{sgn}\begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & n \\ i_1 & \cdots & i_p & i_{p+1} & \cdots & i_n \end{pmatrix} &= (-1)\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & p & p+1 & \cdots & n \\ i_1 & i_2 & \cdots & i_{p+1} & i_p & \cdots & i_n \end{pmatrix} \\
&\vdots \\
&= (-1)^p \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & p+1 & p+2 & \cdots & n \\ i_{p+1} & i_1 & \cdots & i_p & i_{p+2} & \cdots & i_n \end{pmatrix} \\
&= (-1)^{2p} \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & \cdots & p+2 & p+3 & \cdots & n \\ i_{p+1} & i_{p+2} & i_1 & \cdots & i_p & i_{p+3} & \cdots & i_n \end{pmatrix} \\
&\vdots \\
&= (-1)^{(n-p)p} \operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & \cdots & n-p & n-p+1 & \cdots & n \\ i_{p+1} & i_{p+2} & i_{p+3} & \cdots & i_n & i_1 & \cdots & i_p \end{pmatrix}
\end{aligned}$$

b) The map $\Lambda^p V \times \Lambda^p V \to \mathbf{R}$ defined by $(\alpha, \beta) \mapsto \star(\alpha \wedge \star\beta)$ is an inner product on $\Lambda^p V$, with respect to which (for $p = 1$), $\{v_1, \ldots, v_n\}$ is orthonormal.

Solution: First we give the map a name. Define $g(\alpha, \beta) = \star(\alpha \wedge \star\beta)$.

i) For $\alpha, \beta \in \Lambda^p V$, the map g is the composition of a linear map $(\alpha, \beta) \to (a, \star\beta)$, a bilinear map $(\alpha, \gamma) \mapsto \alpha \wedge \gamma$, and finally a linear map $\delta \to \star\delta$ (where $\delta = \alpha \wedge \star\beta$). Thus, g is bilinear.

ii) We show g is symmetric. By (i), it suffices to do this on the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n\}$. Indeed, we show that this basis is 'orthonormal' for g. (I put the inverted commas there since we usually only speak of orthonormality once we know we have an inner product. At this point we do not know g is an inner product.) That, we will show that

$$g(e_{i_1} \wedge \cdots \wedge e_{i_p}, e_{j_1} \wedge \cdots \wedge e_{j_p}) = \delta_{i_1 j_1} \ldots \delta_{i_p j_p}.$$

Note first that given $1 \le i_1 < \cdots < i_p \le n$ and $1 \le j_1 < \cdots < j_p \le n$ (and the corresponding $1 \le i_{p+1} < \cdots < i_n \le n$ and $1 \le j_{p+1} < \cdots < j_n \le n$), that

a) $e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_{p+1}} \wedge \ldots \wedge e_{j_n} = \delta_{i_1 j_1} \ldots \delta_{i_p j_p} e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{i_{p+1}} \wedge \ldots \wedge e_{i_n}$,

b) $\star(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{i_{p+1}} \wedge \ldots \wedge e_{i_n}) = \varepsilon_{i_1, \ldots, i_p} \star (e_1 \wedge \cdots \wedge e_n) = \varepsilon_{i_1, \ldots, i_p}$.

Then,

$$\begin{aligned}
g(e_{i_1} \wedge \cdots \wedge e_{i_p}, e_{j_1} \wedge \cdots \wedge e_{j_p}) &= \varepsilon_{j_1, \ldots, j_p} \star (e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{j_{p+1}} \wedge \ldots \wedge e_{j_n}) \\
&= \varepsilon_{j_1, \ldots, j_p} \delta_{i_1 j_1} \ldots \delta_{i_p j_p} \star (e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{i_{p+1}} \wedge \cdots \wedge e_{i_n}) \\
&= \varepsilon_{i_1, \ldots, i_p} \delta_{i_1 j_1} \ldots \delta_{i_p j_p} \varepsilon_{i_1, \ldots, i_p} \\
&= \delta_{i_1 j_1} \ldots \delta_{i_p j_p},
\end{aligned}$$

as claimed. Thus the matrix of g w.r.t. the ordered basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n\}$ (ordered lexicographically) is indeed the identity matrix of size $\binom{n}{p}$, which is indeed symmetric. Hence, g is symmetric.

iii) We now show g is positive definite. Note that while we could have checked (ii) on *any* basis of $\Lambda^p V$, (since $g(a^i v_i, b^j v_j) = a^i b^j g(v_i, v_j) = a^i b^j g(v_j, v_i) = g(b^j v_j, a^i v_i)$), it is **not** enough to show that "$g(v_i, v_i) \ge 0$ and $g(v_i, v_i) = 0 \iff v_i = 0$" for *any* basis.[1]

[1] Here's an example. Consider the standard ordered basis $\{e_1, e_2\}$ of $\mathbf{R}^2$, and suppose $[g(e_i, e_j)] = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$. Then g satisfies $g(e_i, e_i) \ge 0$ and $g(e_i, e_i) = 0 \iff e_i = 0$ for this basis, but indeed if $v = e_1 + e_2$, then $g(v, v) = -2$!

However, we *can* check (iii) on any basis $\{v_i \mid 1 \le i \le \binom{n}{p}\}$ of $\Lambda^p V$ satisfying $g(v_i, v_j) = \delta_{ij}$, and here's why: Let $\alpha \in \Lambda^p V$ and (using the Einstein summation convention) write $\alpha = a^i v_i$ for scalars a^i. Then, $g(v,v) = g(a^i v_i, a^j v_j) = a^i a^j g(v_i, v_j) = a^i a^j \delta_{ij} = \sum_i (a^i)^2$. This shows that $g(v,v) \ge 0$ and that equality holds iff $v = 0$.

Now, as part of (ii), we showed we had such a basis of $\Lambda^p V$ exists! So g is indeed positive definite.

c) Show that $\langle \star\alpha, \star\beta \rangle = \langle \alpha, \beta \rangle$, for all $\alpha, \beta \in \Lambda^p V$, where the inner product is that from (b).

Solution:

$$\begin{aligned}
\langle \star\alpha, \star\beta \rangle &= \star(\star\alpha \wedge \star\star\beta) \\
&= (-1)^{p(n-p} \star (\star\alpha \wedge \beta) \\
&= (-1)^{p(n-p)}(-1)^{p(n-p)} \star (\beta \wedge \star\alpha) \\
&= \langle \beta, \alpha \rangle \\
&= \langle \alpha, \beta \rangle, \qquad \text{by (ii).}
\end{aligned}$$

33. Prove that $\mathbf{S}^2$ is a smooth 2 dimensional manifold, by using the stereographic projections from the north and south poles as chart maps. (b) Also prove that $\mathbf{S}^2$ is path connected (use smooth paths). (c) Prove that $\mathbf{S}^2$ has no atlas with just 1 chart.

Solution:

Lemma 1. *(i) If U is open in $\mathbf{R}^3$ and contains $\mathbf{S}^2$, and $f : U \to \mathbf{R}^m$ is continuous, then the restriction $\tilde{f}$ of f to $\mathbf{S}^2$ is also continuous. (ii) If V is open in $\mathbf{R}^m$, $g : V \to \mathbf{R}^3$ is continuous and $g(V) \subseteq \mathbf{S}^2$, then $g : V \to \mathbf{S}^2$ is also continuous.*

Proof of Lemma 1. (i) Let $W \subseteq R^m$ be open (in $\mathbf{R}^m$). Since $f : U \to \mathbf{R}^m$ is continuous, $f^{-1}(W)$ is open in $\mathbf{R}^3$. But then $\mathbf{S}^2 \cap f^{-1}(W) = \tilde{f}^{-1}(W)$ is open in $\mathbf{S}^2$. Hence $\tilde{f}$ is continuous.
(ii) Let $X \subseteq \mathbf{S}^2$ be open in $\mathbf{S}^2$. Then $X = U \cap \mathbf{S}^2$ for some open set $U \subseteq \mathbf{R}^3$. Since g is cts, $g^{-1}(U)$ is open in V (and so in $\mathbf{R}^m$). As $g(V) \subseteq \mathbf{S}^2$, $g^{-1}(X) = g^{-1}(U) \cap \mathbf{S}^2$. Hence $g : V \to \mathbf{S}^2$ is also continuous. □

Let p_N, p_S denote the stereographic projections from the north and south poles respectively. A short computation shows that

$$\begin{aligned}
p_N(x,y,z) &= \frac{(x,y)}{(1-z)}, \quad \text{for } (x,y,z) \in \mathbf{S}^2 \setminus \{N\}, \\
p_S(x,y,z) &= \frac{(x,y)}{(1+z)}, \quad \text{for } (x,y,z) \in \mathbf{S}^2 \setminus \{S\}, \\
p_N^{-1}(u,v) &= \frac{(2u, 2v, u^2+v^2-1)}{u^2+v^2+1}, \quad \forall (u,v) \in \mathbf{R}^2, \text{ and} \\
p_S^{-1}(u,v) &= \frac{(2u, 2v, 1-u^2-v^2)}{u^2+v^2+1}, \quad \forall (u,v) \in \mathbf{R}^2.
\end{aligned}$$

Moreover, if $(u,v) \in \mathbf{R}^2 \setminus \{(0,0)\}$ we find $p_S \circ p_N^{-1}(u,v) = \dfrac{(u,v)}{u^2+v^2} = p_N \circ p_S^{-1}(u,v)$. Note first that $\mathbf{S}^2 \setminus \{N\} = \mathbf{S}^\cap \{w \in \mathbf{R}^3 \mid z(w) < 1\}$ and $\mathbf{S}^2 \setminus \{S\} = \mathbf{S}^\cap \{w \in \mathbf{R}^3 \mid z(w) > -1\}$ are both open in $\mathbf{S}^2$. In addition, with the first group of formulae above and Lemma 1 above, we see that

a) p_N, p_N^{-1}, p_S, and p_s^{-1} are continuous,
b) $p_N(\mathbf{S}^2 \setminus \{N\}) = \mathbf{R}^2$ is open in $\mathbf{R}^2$, and
c) $p_S \mathbf{S}^2 \setminus \{S\}) = \mathbf{R}^2$ is open in $\mathbf{R}^2$.

Hence $\{(p_N, \mathbf{S}^2 \setminus \{N\}), ((p_S, \mathbf{S}^2 \setminus \{S\})\}$ is a collection of charts on $\mathbf{S}^2$. Finally, the last two formulae above show that the transition functions $p_S \circ p_N^{-1}$ and $p_N \circ p_S^{-1}$ are both smooth on their (common) domain, namely $\mathbf{R}^2 \setminus \{(0,0)\}$. Thus the smooth atlas $\{(p_N, \mathbf{S}^2 \setminus \{N\}), ((p_S, \mathbf{S}^2 \setminus \{S\})\}$ demonstrates that $\mathbf{S}^2$ is a smooth 2 dimensional manifold.

Now suppose $P, Q \in \mathbf{S}^2$ and suppose at least one of these is not in $\{N, S\}$. If (say) $P \notin \{N, S\}$, and $Q \ne N$, then the path $\beta : [0,1] \to \mathbf{S}^2$ defined by $\beta(t) = p_N^{-1}((1-t)p_N(P) + t p_N(Q))$, being the composition of a linear map with the smooth map p_N^{-1} is clearly smooth as a map into $\mathbf{S}^2$, since $p_N \circ \beta$ is smooth as a map from $[0,1] \to \mathbf{R}^2$. It satisfies $\beta(0) = P, \beta(1) = Q$. If $P \notin \{N, S\}$ and $Q = S$, replace p_N by p_S in the above formula. Finally if $\{N, S\} = \{P, Q\}$, use the stereographic projection from (say) $E = (1,0,0)$ (to the plane with equation $x = 0$), and mimic the formulae above. Note that we could simply have chosen a point R on $\mathbf{S}^2$ which is neither P nor Q, and used the stereographic projection from R, to the plane $\{v \in \mathbf{R}^3 \mid v \cdot R = 0\}$, which is isomorphic to $\mathbf{R}^2$.

Alternatively, if you don't like stereographic projections from points other than N or S, note that if we define $\gamma : [0, \pi] \to \mathbf{R}^3$ by $\gamma(t) = (\sin t, 0, \cos t)$, then γ is certainly a smooth map into $\mathbf{R}^3$ whose image lies in $\mathbf{S}^2$, but it is not immediately clear that γ is a smooth map into $\mathbf{S}^2$. This can be checked in this case via charts covering N and S, or remedied with the (once and for all) lemma:

Lemma 2. *If N is a smooth manifold and $f : N \to \mathbf{S}^2$ is smooth when composed with the inclusion map $i : \mathbf{S}^2 \to \mathbf{R}^3 \setminus \{0\}$, then f is a smooth map into $\mathbf{S}^2$.*

Proof of lemma 2.

Define $p : \mathbf{R}^3 \setminus \{0\} \to \mathbf{S}^2$ by $p(v) = \frac{v}{\|v\|}$. We claim that p is a smooth map.

Cover the manifold $\mathbf{R}^3 \setminus \{0\}$ with the open sets $U_N = \mathbf{R}^3 \setminus \{(0, 0, t) \mid t \geq 0\}$ and $U_S = \mathbf{R}^3 \setminus \{(0, 0, -t) \mid t \geq 0\}$, and use $\varphi_N : U_N \to \mathbf{R}^3$, $\varphi_S : U_S \to \mathbf{R}^3$ defined by $\varphi_{\pm N}(v) = v$ as charts. Note that $\varphi_{\pm N}(U_{\pm N}) = U_{\pm N}$. (Don't mix the pluses and minuses on different sides of the equation here, or in what follows). Clearly, $\{(\varphi_N, U_N), (\varphi_S, U_S)\}$ is a smooth atlas for $\mathbf{R}^3 \setminus \{0\}$, as the transition functions are also the identity on their domains.

To see that p is smooth, it then suffices to check that $p_{\pm N} \circ p \circ \varphi_{\pm}^{-1} N$ are smooth as maps from $U_{\pm N} \to \mathbf{R}^2$. But if $(x, y, z) \in U_N$,

$$p_N \circ p \circ \varphi_N^{-1}(x, y, z) = \frac{(x, y)}{\sqrt{x^2 + y^2 + z^2} - z},$$

and the denominator is only zero when $x = y = 0$, points which are excluded from U_N. A similar argument show that $p_N \circ p \circ \varphi_S^{-1}$ is smooth. Hence, p is smooth.

But then, $f = p \circ (i \circ f)$ is a composition of smooth maps and hence is smooth. $\square$

A third possibility is to use the first case, and appropriate reflections in planes through the origin in $\mathbf{R}^3$. These will restrict to maps $\mathbf{S}^2 \to \mathbf{S}^2$ and will be smooth by Lemma 2.

c) Suppose $\{(\varphi, \mathbf{S}^2)\}$ were an atlas for $\mathbf{S}^2$. Then the (non-empty) set $V = \varphi(\mathbf{S}^2)$ must be open in $\mathbf{R}^2$. Now, we know from second year analysis that that $\mathbf{S}^2$, as a subset of $\mathbf{R}^3$, is compact. We know the continuous image of a compact set is compact, so V is compact in $\mathbf{R}^2$. Thus, V is closed and bounded. Indeed, V is non-empty, open, closed and bounded. We obtain a contradiction directly from the following (as $\mathbf{R}^n$ is not bounded):

Lemma 3. *The only open and closed sets in $\mathbf{R}^n$ are the empty set and $\mathbf{R}^n$.*

Proof of Lemma 3. Suppose $W \subset \mathbf{R}^n$ is open and closed, and neither empty nor $\mathbf{R}^n$. Thus both W and W^c are non-empty. Choose $a \in W$ and $b \in W^c$, and define $\gamma : [0, 1] \to \mathbf{R}^n$ by $\gamma(t) = a + t(b - a)$. Then γ is continuous, $\gamma(0) = a \in W$, and $\gamma(1) = b \in W^c$. Let $s = \sup\{t \in [0, 1] \mid \gamma(t) \in W\}$. (This supremum clearly exists.) From second year, we know that as W is closed, and γ is continuous, $\gamma(s) \in W$. Hence $s < 1$. But W is open, and $\gamma(s) \in W$, so $\exists r > 0$ such that $\mathbf{B}(\gamma(s), r) \subset W$. Let $r_0 = \min\{1 - s, \dfrac{r}{\|b - a\|}\} > 0$. But then, $\gamma(s + r_0) \in \mathbf{B}(\gamma(s), r) \subset W$.

This contradicts $s = \sup\{t \in [0, 1] \mid \gamma(t) \in W\}$, and so W is empty or W is $\mathbf{R}^n$. $\square$

34. Rossmann's exercise 1.2: 8 (modified) Let M be the set $\mathbf{R}$ with the usual topology. Give $\mathbf{R}$ the atlas $\{(\mathrm{id}_R, \mathbf{R})\}$ and denote the corresponding smooth manifold M_0 (this is the usual smooth manifold '$\mathbf{R}$').

a) Define $\varphi : M \to \mathbf{R}$ by $\varphi(t) = t^3$.
   i) Show that $\{(\varphi, M)\}$ is an atlas for a manifold structure on M.
   ii) Is $f = \mathrm{id}_{\mathbf{R}} : M \to M_0$ a diffeomorphism?
   iii) Viewing f as a map $f : M \to M$, is f smooth? (this is what I intended. Sorry for any confusion.)
   iv) Is $g = \mathrm{id}_{\mathbf{R}} : M_0 \to M$ a C^∞ (i.e. smooth) map?
   v) Can you find a diffeomorphism $h : M_0 \to M$? If so, exhibit one.

Solution: (i) M is open in M, and $\varphi : M = \mathbf{R} \to \mathbf{R}$ is surjective, so $\varphi(M) = \mathbf{R}$ is open in $\mathbf{R}$, and indeed is continuous, and has the continuous inverse $\varphi^{-1}(s) = s^{\frac{1}{3}}$. Moreover, M covers M, and the only transition function to check is $\varphi \circ \varphi^{-1} = \mathrm{id}_{\mathbf{R}}$, which is clearly smooth (as a function from $\mathbf{R} \to \mathbf{R}$, the '$\mathbf{R}$' here being M_0).

(ii) Is $f = \mathrm{id}_{\mathbf{R}} : M \to M_0$ a diffeomorphism?

Solution: No: $f = \mathrm{id}_{\mathbf{R}} : M \to \mathbf{R}$ is not even smooth, since $f \circ \varphi^{-1}$ is the map $\varphi^{-1} : s \mapsto s^{\frac{1}{3}}$, which is not even differentiable at $s = 0$.

iii) Viewing f as a map $f : M \to M$, is f smooth? (This is what I intended. Sorry for any confusion.)

Yes, since $\varphi \circ f \circ \varphi^{-1}$ is $s \mapsto s$, which is smooth.

(iv) Is $g = \mathrm{id}_{\mathbf{R}} : M_0 \to M$ a C^∞ (i.e. smooth) map? Yes, since $\varphi \circ g \circ \mathrm{id}_{\mathbf{R}}$ is $s \mapsto s^3$, which is smooth.

(v) Can you find a diffeomorphism $h : M_0 \to M$? If so, exhibit one. If not, explain why.

Solution: Yes: define $h : M_0 \to M$ by $h(s) = s^{\frac{1}{3}}$. Then, h is smooth because (checking on charts at both 'ends') $\varphi \circ h \circ \mathrm{id}_{\mathbf{R}}^{-1}$ is the map $s \mapsto s$, which is a smooth map in the usual sense: as a function from $\mathbf{R} \to \mathbf{R}$, no tricks. Moreover, the inverse of h, $h^{-1} : M \to M_0$, defined by $h^{-1}(s) = s^3$, is also smooth, as (checking again on charts at both 'ends') $\mathrm{id}_{\mathbf{R}} \circ h^{-1} \varphi^{-1}$ is again the map $s \mapsto s$, which is a smooth map in the usual sense: as a function from $\mathbf{R} \to \mathbf{R}$, again, no tricks.

New. (Warner, P. 10) Define $f, g, h : \mathbf{R} \to \mathbf{R}$ by

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}, \quad g(t) = \frac{f(t)}{f(t) + f(1-t)}, \quad \text{and } h(t) = g(t+2)g(2-t).$$

You may assume that $f \in \mathbf{C}^\infty(\mathbf{R})$.

a) Show that the function h satisfies $h \in C^\infty(\mathbf{R})$, $\forall t,\ |t| \leq 1 \Rightarrow h(t) = 1$ and $\forall t,\ |t| > 2 \Rightarrow h(t) = 0$.

Solution: This is a straightforward check.

b) Let $r > 0$. Find a function $k \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\forall v$, $\|v\| \leq r \Rightarrow k(v) = 1$, and $\forall v$, $\|v\| > 2r \Rightarrow k(v) = 0$.

Solution: Define $k(v) = h\Big(\frac{\|v\|}{r}\Big)$.

Then k clearly satisfies all conditions except possibly smoothness at $v = 0$. But inside the open set $B(0, r) = \{v \mid \|v\| < r\}$, k is identically 1, and so is clearly smooth at $v = 0$.

c) Let $r > 0$. Find a function $l \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\forall v$, $\|v\| \leq r \Rightarrow l(v) = 0$, and $\forall v$, $\|v\| > 2r \Rightarrow l(v) = 1$.

Solution: Set $l = 1 - k$.

d) Suppose $f \in C^\infty(\mathbf{S}^2)$. Prove that there is $g \in \mathbf{C}^\infty(\mathbf{R}^3)$ such that $g(v) = f(v), \forall v \in \mathbf{S}^2$. (Hint: Use the map $v \mapsto \frac{v}{\|v\|}$, f, and part (c) to fix things at $v = 0$.)

Solution: Given $f \in C^\infty(\mathbf{S}^2)$, let $l \in \mathbf{C}^\infty(\mathbf{R}^3)$ be the function designed in part (c) (for $r = \frac{1}{4}$), and define $g : \mathbf{R}^3 \to \mathbf{R}$ by

$$g(v) = \begin{cases} l(v) f\Big(\frac{v}{\|v\|}\Big), & v \neq 0 \\ 0, & v = 0 \end{cases}$$

Then g is clearly smooth except possibly at $v = 0$. However, (see part (c)), g is identically 0 inside the open set $B(0, \frac{1}{4})$, and so is now clearly smooth at $v = 0$. Note that for all v with $\|v\| > \frac{1}{2}$, $l(v) = 1$, and so if $v \in \mathbf{S}^2$, i.e. $\|v\| = 1$, $g(v) = f(v)$. In other words, $f = g \circ i$.

e) Show that $i_* : T_p\mathbf{S}^2 \to T_p\mathbf{R}^3$ is an injective linear map. Prove that i_* is an injective linear map. (Hint: Use (d).)

Solution: Let $v_p \in T_p\mathbf{S}^2$ and suppose $i_*(v_p) = 0$. This means that $i_*(v_p)(g) = 0$, for all $g \in C^\infty(\mathbf{R}^3)$. That is, $v_p(g \circ i) = 0$ for all $g \in C^\infty(\mathbf{R}^3)$. But then, given any $f \in C^\infty(\mathbf{S}^2)$, by (d), there is $g \in C^\infty(\mathbf{R}^3)$ with $f = g \circ i$. Hence $v_p(f) = 0$ for all $f \in C^\infty(\mathbf{S}^2)$. Thus, $v_p = 0$, and so i_* is injective.