Tensor Analysis Fall 2017 Assignment 2 (Final) (Due Friday 13-October 4 pm)

Question numbers refer to those on the list on the web. Make no assumptions on dimension here, unless otherwise specified. All responses require sufficient justification.

24. Let $\{v_1, \dots, v_n\}$ be an ordered orthonormal basis of the inner product space V, and suppose that $\star : \Lambda V \to \Lambda V$ is the Hodge star map associated to the given ordered basis. Show that

a) $\star\star : \Lambda^p V \to \Lambda^p V$ is multiplication by ± 1, and find the exact dependence of the sign on p and n.
b) $\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta)$ for all $\alpha, \beta \in \Lambda^p V$.
c) $\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle$, for all $\alpha, \beta \in \Lambda^p V$.

26. Let V be a vector space of dimension n.

a) Show that $\{u_1, \dots, u_k, v_1, \dots v_k\}$ is linearly independent iff $a = \sum_{i=1}^k u_i \wedge v_i$ satisfies $a^k \neq 0$.
e) Prove that if $v_1 \wedge v_2 \wedge \dots \wedge v_k$ and $w_1 \wedge w_2 \wedge \dots \wedge w_k$ are non-zero rank–one elements of $\Lambda^k V$, then

$$\exists \lambda \neq 0 \text{ s.t. } v_1 v_2 \dots v_k = \lambda w_1 w_2 \dots w_k \iff \operatorname{span}\{v_1, v_2, \dots, v_k\} = \operatorname{span}\{w_1, w_2, \dots, w_k\}.$$

33. a) Prove that $\mathbf{S}^2$ is a connected 2 dimensional smooth manifold by using the stereographic projections from the north and south poles as chart maps.

b) Prove that $\mathbf{S}^2$ is path connected via smooth paths.
c) Prove that $\mathbf{S}^2$ has no atlas with just 1 chart.

34. Rossmann's exercises 1.2: 8 (modified) Let M be the set $\mathbf{R}$ with the usual topology. Give $\mathbf{R}$ the atlas $\{(\mathrm{id}_R, \mathbf{R})\}$ and denote the corresponding manifold M_0 (this is the usual manifold '$\mathbf{R}$').

a) Define $\varphi : M \to \mathbf{R}$ by $\varphi(t) = t^3$.
   i) Show that $\{(\varphi, M)\}$ is an atlas for a manifold structure on M.
   ii) Is $f = \mathrm{id}_{\mathbf{R}} : M \to M_0$ a diffeomorphism?
   iii) Viewing f as a map $f : M \to M$, is f in $C^\infty(M)$?
   iv) Is $g = \mathrm{id}_{\mathbf{R}} : M_0 \to M$ a C^∞ map?
   v) Can you find a diffeomorphism $h : M_0 \to M$? If so, exhibit one.

New. (Warner, P. 10) Define $f, g, h : \mathbf{R} \to \mathbf{R}$ by

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}, \quad g(t) = \frac{f(t)}{f(t) + f(1-t)}, \quad \text{and } h(t) = g(t+2)g(2-t).$$

You may assume that $f \in \mathbf{C}^\infty(\mathbf{R})$.

a) Show that the function h satisfies $h \in C^\infty(\mathbf{R})$, $\forall t,\ |t| \leq 1 \Rightarrow h(t) = 1$ and $\forall t,\ |t| > 2 \Rightarrow h(t) = 0$.
b) Let $r > 0$. Find a function $k \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\forall v,\ \|v\| \leq r \Rightarrow k(v) = 1$, and $\forall v,\ \|v\| > 2r \Rightarrow k(v) = 0$.
c) Let $r > 0$. Find a function $l \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\forall v,\ \|v\| \leq r \Rightarrow l(v) = 0$, and $\forall v,\ \|v\| > 2r \Rightarrow l(v) = 1$.
d) Suppose $f \in C^\infty(\mathbf{S}^2)$. Prove that there is $g \in \mathbf{C}^\infty(\mathbf{R}^3)$ such that $g(v) = f(v), \forall v \in \mathbf{S}^2$. (Hint: Use the map $v \mapsto \frac{v}{\|v\|}$, f, and part (c) to fix things at $v = 0$.)
e) Assuming that that the inclusion map $i : \mathbf{S}^2 \to \mathbf{R}^3$ is smooth map, prove that $i_* : T_p\mathbf{S}^2 \to T_p\mathbf{R}^3$ is an injective linear map. (Hint: Use (d).) (Recall: for $g \in \mathbf{C}^\infty(\mathbf{R}^3)$, $i_*(v_p)(g) := v_p(g \circ i)$)