Faculty of Science »
Department of mathematics and statistics »

MAT 4183 Final Exam

December 16, 2017. Duration: 3 hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY:

1. You have three hours to complete this exam.
2. **This is a closed book exam, and no notes of any kind are permitted. Calculators are not allowed, and the use or possesion on your person of cell phones is not permitted. <u>By signing the attendance sheet you are agreeing that you will comply with these rules.</u>**
3. The correct answer requires reasonable justification written legibly and logically.
4. All questions are worth an equal number of points, and you should aim to finish 6 questions. Read through all of the questions before beginning.
5. *Please* begin each new question on a new page. It will help me and won't waste too much paper if you only use the backs of pages for rough work. If you need more scrap paper, please ask.
6. Bonne chance! Good luck!

Possibly useful formulae:

In the following, ∂_i will denote $\dfrac{\partial}{\partial z^i}$, and ∂_θ will denote $\dfrac{\partial}{\partial \theta}$ for any coordinate function θ.

A. $\star(e^{i_1}\wedge\cdots\wedge e^{i_p}) = \epsilon_{i_1}\cdots\epsilon_{i_p}\,\mathrm{sgn}(i_1,\ldots,i_n)\,e^{i_{p+1}}\wedge\cdots\wedge e^{i_n}$, where $g(e^i,e^j)=\epsilon_i\delta_{i\,j}$

B. $i_*\,\partial_\varphi = \cos\theta\cos\varphi\,\partial_x + \sin\theta\cos\varphi\,\partial_y - \sin\varphi\,\partial_z,\qquad i_*\,\partial_\theta = -\sin\theta\sin\varphi\,\partial_x + \cos\theta\sin\varphi\,\partial_y$

C. $2g(\nabla_u v, w) = u\,g(v,w) - w\,(g(u,v)) + v\,g(w,u) + g([u,v],w) - g(u,[v,w]) + g([w,u],v)$

D. $2\,\Gamma^k_{i\,j} = g^{k\,l}(\partial_i g_{j\,l} - \partial_l g_{i\,j} + \partial_j g_{l\,i})$

E. $\Gamma^\varphi_{\theta\,\theta} = -\sin\varphi\cos\varphi,\qquad \Gamma^\theta_{\theta\,\varphi} = \Gamma^\theta_{\varphi\,\theta} = \cot\varphi$

1. Let V be a finite dimensional real vector space and V^* its dual. Suppose g is a symmetric non-degenerate bilinear form on V, and denote $\psi : V \to V^*$ the isomorphism defined by $\psi(v)(w) = g(v,w)$, $\forall v, w \in V$. Define maps

$$c : V \times V^* \to \mathbf{R} \quad \text{and } \tilde{c} : V \times V^* \to \mathbf{R}$$

by

$$\begin{aligned} c(v,f) &= f(v), \quad \text{and} \\ \tilde{c}(v,f) &= g(v, \psi^{-1}(f)) \quad \text{for all } v \in V, f \in V^*. \end{aligned}$$

a) Explain briefly why there is an element $C \in (V \otimes V^*)^*$ which satisfies

$$C(v \otimes f) = c(v,f), \quad \forall v \in V, f \in V^*.$$

b) There is a natural isomorphism $e : V \otimes V^* \to \mathrm{Hom}\,(V,V)$. Give a brief description of the definition of e.

c) Prove that if $T \in \mathrm{Hom}\,(V,V)$, then $C \circ e^{-1}(T)$ is the trace of the linear transformation T.

Solution: Let $\{v_1, \ldots v_n\}$ be a basis for V and $\{v^1, \ldots, v^n\}$ its dual basis. Then write $T(v_i) = A^i_j v_j$ for scalars A^i_j. Then $e^{-1}(T) = A^i_j v_j \otimes v^i$, so $C(e^{-1}(T)) = A^i_j v^i(v_j) = A^i_j \delta^i_j = A^i_i = \mathrm{tr}(T)$.

d) Prove that $c(v,f) = \tilde{c}(v,f)$ for all $v \in V, f \in V^*$.

Solution:

$$\begin{tikzcd} V \otimes V^* \arrow[d, "\mathrm{id}_V \otimes \psi_g^{-1}"'] \arrow[dr, "C \circ e^{-1}"] & \\ V \otimes V \arrow[r, "g"] & \mathbf{R} \end{tikzcd}$$

Down, then right: $(v,f) \mapsto (v, \psi_g^{-1}(f)) \to g(v, \psi_g^{-1}(f))$ while $(v,f) \mapsto f(v)$. If $w = \psi_g^{-1}(f)$, then $f(v) = \psi_g(w)(v) = g(v,w) = g(v, \psi_g^{-1}(f))$

2. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ and define $T \in \mathrm{Hom}\,(\mathbf{R}^3, \mathbf{R}^3)$ by $T(v) = Av$.

Let $\mathbf{R}^3 \otimes \mathbf{R}^3 = \mathrm{span}\{e_i \otimes e_j \mid 1 \leq i, j \leq 3\}$ be the usual tensor product, and note that $f : \mathbf{R}^3 \otimes \mathbf{R}^3 \to \mathbf{M}_{33}(\mathbf{R})$ defined by $f(v \otimes w) = vw^t$ is an isomorphism.

Recall the natural isomorphism $e : \mathbf{R}^3 \otimes (\mathbf{R}^3)^* \to \mathrm{Hom}\,(\mathbf{R}^3, \mathbf{R}^3)$ and set $t = e^{-1}(T)$.

a) Find an explicit expression for $f^{-1}(A) \in \mathbf{R}^3 \otimes \mathbf{R}^3$.

b) Find an explicit expression for $t \in \mathbf{R}^3 \otimes (\mathbf{R}^3)^*$.

c) Write $t = \sum_{i=1}^{m} v_i \otimes w^i$ for $v_i \in \mathbf{R}^3, w^i \in (\mathbf{R}^3)^*$, where $m = \mathrm{rank}(t)$.

Solution:

a) [3] If $w \in \mathbf{R}^3$,note that $f(e_i \otimes w)$ is the matrix whose ith row is w^t, and which has zeros elsehwere. Thus we have

$$f^{-1}(A) = e_1 \otimes (e_1 + 2e_2 - e_3) \; + \; e_2 \otimes (-e_2 + e_3) \; + \; e_3 \otimes (2e_1 + e_2 + e_3).$$

b) [3] If $v \in \mathbf{R}^3$, then $e(v \otimes e^j)$ is the linear map whose matrix has v as its jth column, and zeros elsewhere. Hence

$$e^{-1}(T) = (e_1 + 2e_3) \otimes e^1 \; + \; (2e_1 - e_2 + e_3) \otimes e^2 \; + \; (-e_1 + e_2 + e_3) \otimes e^3.$$

Alternatively, let $h : \mathbf{R}^3 \otimes \mathbf{R}^3 \to \mathbf{R}^3 \otimes (\mathbf{R}^3)^*$ denote the isomorphism satisfying $h(e_i \otimes e_j) = e_i \otimes e^j$, and $S : \mathrm{Hom}\,(\mathbf{R}^3, \mathbf{R}^3) \to \mathbf{M}_{33}(\mathbf{R})$ the isomorphism which sends a linear map to its standard matrix. Then it is clear that $f = S \circ e \circ h$, and so $e^{-1} = h \circ f^{-1} \circ S$. Then using (a) one also check sees that

$$e^{-1}(T) = e_1 \otimes (e^1 + 2e^2 - e^3) \; + \; e_2 \otimes (-e^2 + e^3) \; + \; e_3 \otimes (2e^1 + e^2 + e^3).$$

c) [4] If we write A in block row form as $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$, a two-step row reduction shows $\mathrm{rank}\, A = \mathrm{rank}\, t = 2$ and, in particular that $r_3 = 2r_1 + 3r_2$. Thus by the second solution to (b) we have

$$e^{-1}(T) = (e_1 + 2e_3) \otimes (e^1 + 2e^2 - e^3) \; + \; (e_2 + 3e_3) \otimes (-e^2 + e^3).$$

Alternatively, noting that the third column of A is the first minus the second, using the first soution to (b) we see that

$$e^{-1}(T) = (e_1 + 2e_3) \otimes (e^1 + e^3) \; + \; (2e_1 - e_2 + e_3) \otimes (e^2 - e^3).$$

Or, noting that the first column of A is the sum of the second and third, using the first soution to (b) we see that

$$e^{-1}(T) = \; (2e_1 - e_2 + e_3) \otimes (e^1 + e^2) \; + \; (-e_1 + e_2 + e_3) \otimes (e^1 + e^3).$$

3. Let (V_1, g_1) and (V_2, g_2) be semi-Riemannian vector spaces of the same dimension. (So, g_1 and g_2 are symmetric, non-degenerate bilinear forms on V_1 and V_2 respectively.) Recall that a subset $\{u_1, \dots, u_n\} \subset V_1$ is *orthonormal* if $g_1(u_i, u_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = \pm 1$.

A map $T : V_1 \to V_2$ is an *isometry* (of semi-Riemannian vector spaces) if

$$g_1(u, v) = g_2(Tu, Tv), \quad \forall u, v \in V_1.$$

a) Define "g_1 is non-degenerate."

b) Prove that if $\{u_1, \dots, u_n\} \subset V_1$ is an orthonormal basis of V_1, then $\{Tu_1, \dots, Tu_n\} \subset V_2$ is an orthonormal basis of V_2.

(Hint: (i) That $\{Tu_1, \dots, Tu_n\}$ is an orthonormal *set* is trivial. The important part is to show that $\{Tu_1, \dots, Tu_n\}$ is linearly independent – but do not assume that every isometry is a *linear map* unless you have done part (e)!)

Solution: $0 = \sum_i \lambda_i T(u_i) \Rightarrow \forall j, \quad 0 = g_2(\sum_i \lambda_i T(u_i), T(u_j)) = \sum_i \lambda_i g_1(u_i, u_j) = \lambda_j$, and $\dim V_1 = \dim V_2$. (Note that if $u \neq 0$ is a null vector, then $\{u, 2u\}$ is orthogonal, but of course is not l.i..)

c) Suppose $V = \mathbf{R}^4$ and that the matrix of a bilinear form $g : V \times V \to \mathbf{R}$ with respect to the standard ordered basis of $\mathbf{R}^4$ is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Find an orthonormal basis of V.

Solution: Don't try Gram Schmidt, since here every standard basis vector is null! However, using the two-dimensional version as a guide, we see that $\{\frac{\sqrt{2}}{2}(e_1 + e_3), \frac{\sqrt{2}}{2}(e_1 - e_3), \frac{\sqrt{2}}{2}(e_2 + e_4), \frac{\sqrt{2}}{2}(e_2 - e_4)\}$ is orthonormal.

e) Prove that an isometry is a linear map.

(Hint: Consider the expression $g_2(T(u + \lambda v) - T(u) - \lambda T(v), T(u_j))$, where u_j belongs to an orthonormal basis in V_1.)

Solution:

For all u, v, λ, j, we have

$$g_2(T(u + \lambda v) - T(u) - \lambda T(v), T(u_j)) = g_1(u + \lambda v, u_j) - g_1(u, u_j) - \lambda g_1(v, u_j) = 0,$$

because g_1, g_2 are bilinear. Morover, $\forall w \in V_2$, $w = \sum_i \lambda_i T(u_i)$, so

$$\forall j, \quad 0 = g_2(w, T(u_j)) = \sum_i \lambda_i \delta_{ij} = \lambda_i \Rightarrow w = 0.$$

Hence $\forall u, v, \lambda, \quad T(u + \lambda v) - T(u) - \lambda T(v) = 0$. Hence T is linear.

4. a) Let V be a vector space and V^* its dual. For $v \in V$, define a map $i_v : \Lambda^k V^* \to \Lambda^{k-1} V^*$ by

$$i_v(f_1 \wedge \ldots \wedge f_k) = \sum_{j=1}^{k} (-1)^{j-1} f_j(v)\ f_1 \wedge \ldots \wedge \hat{f_j} \wedge \ldots \wedge f_k, \text{ for } k \geq 1,$$

and define i_v on $\Lambda^0 V^*$ to be zero. You may assume without proof that i_v satisfies a graded Leibniz rule: That is,

$$i_v(\alpha \wedge \beta) = i_v\alpha \wedge \beta + (-1)^p \alpha \wedge i_v\beta, \qquad \text{for all } v \in V, \alpha \in \Lambda^p V^*, \beta \in \Lambda V.$$

a) Prove that $i_v^2 : \Lambda^k V^* \to \Lambda^{k-e} V^*$ satisfies the usual Leibniz rule:

$$i_v^2(\alpha \wedge \beta) = i_v^2\alpha \wedge \beta + a \wedge i_v^2\beta.$$

Solution: Use the rule, noting that on the second application , the degree of $i_v\alpha$ is $\deg \alpha - 1$, so like terms cancel, not add.

b) Use (a) to show that $i_v^2 = 0$ on ΛV^*.

Solution: The linear map i_v^2 is zero on $\Lambda^0 V$ and V^*. Since i_v^2 satisfies (1) above, it is zero on all elements of rank 1. But $\Lambda^k V^*$ is spanned by elements of rank 1, so $i_v^2 = 0$ on ΛV^*.

Now let M be a smooth manifold and $v \in \mathrm{Vect}(M)$. Define maps

$$i_v : \Omega^k(M) \to \Omega^{k-1}(M) \text{ and } \mathcal{L}_v : \Omega^k(M) \to \Omega^k(M)$$

respectively by

$$(i_v\alpha)(p) = i_{v_p}(\alpha_p), \text{ and } \mathcal{L}_v(\alpha) = di_v\alpha + i_v d\alpha,$$

where v_p and α_p are v and α evaluated at p respectively, and i_{v_p} is the map defined above for $V = T_pM$.

c) Prove that if $d\alpha = 0$, for some $\alpha \in \Omega^k(M)$, then $d(\mathcal{L}_v\alpha) = 0$.

d) Prove that if $\alpha = d\beta$, for some $\beta \in \Omega^k(M)$, then $\mathcal{L}_v\alpha = d\gamma$ for some $\gamma \in \Omega^k(M)$.

Solution: If $\alpha = d\beta$, then $\mathcal{L}_v\alpha = d(i_v d\beta)$.

e) If $M = \mathbf{R}^2$, $v = a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y}$ for *constant functions* a and b, then $\mathcal{L}_v(dx) = 0$.

Solution: $\mathcal{L}_v(dx) = di_v dx + i_v d(dx) = di_v dx = d\Big(i_{a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}}\Big)(dx) = d(a) = 0$, since i_v is clearly linear in v.

5. Let $\mathbf{S}^1$ denote the circle and $\mathbf{S}^1 \xrightarrow{i} \mathbf{R}^2$ the smooth inclusion map.

a) Define a 1-form on $\mathbf{S}^1$ by $\omega = i^*(-y\ dx + x\ dy)$. Prove that $d\omega = 0$, but that there is no smooth function $\theta : \mathbf{S}^1 \to \mathbf{R}$ such that $\omega = d\theta$.

b) Prove that $\omega_p \neq 0$ for all $p \in \mathbf{S}^1$. (*Hint: Find a smooth curve* $\beta : \mathbf{R} \to \mathbf{S}^1$ *such that (i)* $\operatorname{im} \gamma = \mathbf{S}^1$ *and (ii)* $\omega_{\beta(t)}(\beta'(t)) \neq 0$ *for all* $t \in \mathbf{R}$.)

c) Prove that there is a diffeomorphism $\psi : S^1 \times \mathbf{R} \to \mathrm{T}\mathbf{S}^1$, linear on the fibres, which makes the diagram

$$\begin{tikzcd} \mathbf{S}^1 \times \mathbf{R} \arrow[r, "\psi"] \arrow[d, "p_1"'] & \mathrm{T}\mathbf{S}^1 \arrow[d, "\pi"] \\ \mathbf{S}^1 \arrow[r, "\mathrm{id}_{\mathbf{S}^1}"] & \mathbf{S}^1 \end{tikzcd}$$

commutative iff there is $v \in \mathrm{Vect}(\mathbf{S}^1)$ such that $v_p \neq 0, \forall p \in \mathbf{S}^1$. (Here, $p_1(z, r) = z$, and you need only define your map and show it is bijective. You do not need to prove it is smooth.)

Solution: Suppose there is a nowhere-vanishing $v \in \mathrm{Vect}(\mathbf{S}^1)$, and define $\psi : T\mathbf{S}^1 \to \mathbf{S}^1 \times R$ by $\psi(p, r) = r.v_p$. This is clearly linear on the fibres, and as $\dim T_p\mathbf{S}^1 = 1 = \dim \mathbf{R}$, is an iso there. This shows that ψ is a bijection.

Conversely, suppose such a ψ as above exists, and define $v : \mathbf{S}^1 \to T\mathbf{S}^1$ by $v_p = \psi(p, 1)$. Since ψ is linear on the fibres and is a bijection, ψ restricted to the fibres is injective and thus $v_p = \psi(p, 1) \neq 0$.

d) Prove that there is a vector field $v \in \mathrm{Vect}(\mathbf{S}^1)$ such that $v_p \neq 0, \forall p \in \mathbf{S}^1$.

Solution: Use the β from part (b)!

6. Give $M = \mathbf{R}^2 \setminus \{(0,0)\}$ the smooth structure it inherits as an open subset of $\mathbf{R}^2$. Define $\omega \in \Omega^1(M)$ and $v \in \mathrm{Vect}(M)$ by

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

and

$$v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Now let $\varphi_\lambda : M \to M$ be defined by $\varphi(v) = \lambda v$ for some fixed real number $\lambda > 0$.

a) Define an *integral curve* of a vector field $u \in \mathrm{Vect}(M)$.

b) Define the *flow* $\phi_t : M \to M$ generated by an integrable vector field $u \in \mathrm{Vect}(M)$.

c) Show that v is integrable and that its flow satisfies $\phi_t(p) = \varphi_{e^t}(p), \forall p \in M$.

Solution: Denote $p = (a, b)$ and $\phi_t(p) = (x_p(t), y_p(t))$. Then we solve $\dot{x}_p(t) = x_p(t)$ with $x_p(0) = a$ and $\dot{y}_p(t) = y_p(t)$ with $y_p(0) = b$. These have the solutions $x_p(t) = ae^t$ and $y_p(t) = be^t$, so that $\phi_t(p) = e^t(a, b) = e^t p = \varphi_{e^t}(p)$. The solutions exist for all $t \in \mathbf{R}$ so v is integrable.

d) Prove that $(\varphi_\lambda)^*\omega = \omega$ for all $\lambda > 0$.

Solution: Note that $(\varphi_\lambda)^*(x) = \lambda x$, and $(\varphi_\lambda)^*(y) = \lambda x$, so $(\varphi_\lambda)^*(dx) = d((\varphi_\lambda)^*(x)) = \lambda dx$, and $(\varphi_\lambda)^*(dy) = \lambda dy$. Then,

$$\begin{aligned}
(\varphi_\lambda)^*\omega &= -\frac{(\varphi_\lambda)^* y}{(\varphi_\lambda)^*(x^2+y^2)}(\varphi_\lambda)^* dx + \frac{(\varphi_\lambda)^* x}{(\varphi_\lambda)^*(x^2+y^2)}(\varphi_\lambda)^* dy \\
&= -\frac{\lambda y}{\lambda^2(x^2+y^2)}\lambda dx + \frac{\lambda x}{\lambda^2(x^2+y^2)}\lambda dy \\
&= \omega.
\end{aligned}$$

7. Equip $\mathbf{R}^3 = \{(t,x,y) \mid t,x,y \in \mathbf{R}\}$, with the volume form $\omega = dt \wedge dx \wedge dy$ and the Minkowski metric $g = -dt \otimes dt + dx \otimes dx + dy \otimes dy$.

a) Complete the following table, where $\star$ is the Hodge-star map in this case for the (orthonormal) ordered basis $\{dt, dx, dy\}$.

Solution: Since $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, 1, 1)$,

$\star(dt \wedge dx)$	$-dy$
$\star(dt \wedge dy)$	dx
$\star(dx \wedge dy)$	dt
$\star(dt \wedge dx \wedge dy)$	-1

b) Let $F \in \Omega^2(\mathbf{R}^3)$ be a smooth 2-form on $\mathbf{R}^3$. Show that we may write $F = G + dt \wedge H$, where

$$G = a\,dx \wedge dy \quad \text{and} \quad H = b\,dx + c\,dy, \quad \text{for some } a,b,c \in \mathbf{C}^\infty(\mathbf{R}^3).$$

Solution: Any 2-form F on $\mathbf{R}^3$ is of the form $F = a\,dx \wedge dy + b\,dt \wedge dx + c\,dt \wedge dy = a\,dx \wedge dy + dt \wedge (b\,dx + c\,dy)$ for some $a,b,c \in \mathbf{C}^\infty(\mathbf{R}^3)$, and is clearly of the desired form.

c) Show that the equation $dF = 0$ is equivalent to $\dfrac{\partial a}{\partial t} = \dfrac{\partial c}{\partial x} - \dfrac{\partial b}{\partial y}$

Solution:

$$\begin{aligned}
0 = dF &= dG - dt \wedge dH \\
&= (\partial_t a\ dt + \partial_x a\ dx + \partial_y a\ dy) \wedge dx \wedge dy - dt \wedge (\partial_x b\ dx \wedge dx + \partial_y b\ dy \wedge dx + \partial_t b\ dt \wedge dx) \\
&- -dt \wedge (\partial_x c\ dx \wedge dy + \partial_y c\ dy \wedge dy + \partial_t c\ dt \wedge dy) \\
&= (\partial_t a + \partial_y b - \partial_x c)\ dt \wedge dx \wedge dy
\end{aligned}$$

Since $(dt \wedge dx \wedge dy)_p \neq 0$, $\forall p \in \mathbf{R}^3$, this implies $\partial_t a + \partial_y b - \partial_x c = 0$, and the desired result follows.

d) Show that the equation $d \star F = 0$ is equivalent to

$$\frac{\partial a}{\partial x} - \frac{\partial c}{\partial t} = \frac{\partial a}{\partial y} + \frac{\partial b}{\partial t} = \frac{\partial b}{\partial x} + \frac{\partial c}{\partial y} = 0.$$

Solution: From the table in (a),

$$\begin{aligned}
\star F &= \star(a\ dx \wedge dy) + \star(b\ dt \wedge dx + c\ dt \wedge dy) \\
&= a\ dt - b\ dy + c\ dx,
\end{aligned}$$

so

$$\begin{aligned}
d \star F &= d(a\ dt - b\ dy + c\ dx) \\
&= \partial_x a\ dx \wedge dt + \partial_y a\ dy \wedge dt - \partial_t b\ dt \wedge dy - \partial_x b\ dx \wedge dy + \partial_t c\ dt \wedge dx + \partial_y c\ dy \wedge dx \\
&= (-\partial_x a + \partial_t c) dt \wedge dx + (-\partial_y a - \partial_t b)\ dt \wedge dy + (-\partial_x b - \partial_y c)\ dx \wedge dy
\end{aligned}$$

This equation and the linear independence of $dt \wedge dx$, $dt \wedge dy$ and $dx \wedge dy$ now yields the desired equivalence.

e)[Bonus] Show that the only solutions to $dF = 0 = d \star F$ with $a = e^{t-\lambda(x-y)}$ must satisfy $\lambda^2 = \frac{1}{2}$.

Solution: Indeed, if $a = e^{t-\lambda(x-y)}$, then (denoting $\partial_{x^i} a$ by a_{x^i}, etc.), we have $a_t = a$ and $a_x = -\lambda a = -a_y$.

Then, $a_x = c_t$ and $a_y = -b_t$ implies $b = -\lambda a + \bar{b}$, and $c = -\lambda a + \bar{c}$ where $\bar{b}, \bar{c}$ satisfy $\bar{b}_t = \bar{c}_t = 0$. Then, $a_t = c_x - b_y \iff a = \lambda^2 a + \bar{c}_x - (-\lambda^2 a + \bar{b}_y) \iff (1 - 2\lambda^2)a = \bar{c}_x - \bar{b}_y$. But taking partials w.r.t. t on both sides yields $(1 - 2\lambda^2)a = 0$, whence $\lambda^2 = \frac{1}{2}$.

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8. Let M be a manifold with connection D. For $p \in M$ and $u \in \mathrm{Vect}(M)$, let $u_p \in T_pM$ denote the tangent vector obtained by evaluating the vector field u at $p \in M$.

Suppose that $\omega \in \Omega^1(M)$, and define two functions $D\omega, T\omega : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathbf{C}^\infty(M)$ by

$$\begin{aligned} D\omega(u, v) &= u(\omega(v)) - \omega(D_u v), \quad \forall u, v \in \mathrm{Vect}(M), \text{and} \\ T\omega(u, v) &= u(\omega(v)) - v(\omega(u)) - \omega([u, v]), \quad \forall u, v \in \mathrm{Vect}(M) \end{aligned}$$

a) Prove that $D\omega$ is $\mathbf{C}^\infty(M)$–bilinear.

Solution: Let $f, g \in \mathbf{C}^\infty(M)$. Then

$$\begin{aligned} D\omega(fu, gv) &= fu(\omega(gv)) - \omega(D_{fu} gv) \\ &= fu(g(\omega(v)) - f\omega(D_u gv) \\ &= f\Big(u(g)(\omega(v) + gu(\omega(v))\Big) - f\omega\Big(u(g)v + gD_u v\Big) \\ &= fg\, u(\omega(v)) + fu(g)\omega(v) - fu(g)\omega(v) - fg\,\omega(D_u v) \\ &= fg\, D\omega(u, v) \end{aligned}$$

b) Prove that $T\omega$ is $\mathbf{C}^\infty(M)$–bilinear.

Solution: Let $f, g \in \mathbf{C}^\infty(M)$. If $u, v \in Vect(M)$, first note that an easy computaion shows that

$$[f\, u, v] = f[u, v] - v(f)u.$$

Then

$$\begin{aligned} T\omega(fu, v) &= fu(\omega(v)) - v(\omega(fu)) - \omega([fu, v]) \\ &= fu(\omega(v)) - \Big(v(f)\omega(u) + f\, v(\omega(u))\Big) - \omega\Big(f[u, v] - v(f)u\Big) \\ &= f\, u(\omega(v)) - v(f)\omega(u) - f\, v(\omega(u)) \ - f\,\omega([u, v]) + v(f)\omega(u) \\ &= f\, T\omega(u, v) \end{aligned}$$

Since $T\omega$ is clearly antisymmetric in its arguments, we're done.

c) For $u, v \in \mathrm{Vect}(M)$, show that $(D\omega(u, v))(p)$ depends only on u_p and v_p.

(Hint: Use local coordinates around p and consider the the smooth functions $B_{i\,j}$ defined by $B_{i,j} = D\omega(\partial_i, \partial_j)$.)

Solution: Suppose $u = u^i\partial_i$ and $v = v^j\partial_j$ for smooth functions u^i, v^j. hen, by (b),

$$\begin{aligned} D\omega(u, v) &= D\omega(u^i\partial_i, v^j\partial_j) \\ &= u^i v^j D\omega(\partial_i, \partial_j) \\ &= u^i v^j B_{i\,j} \end{aligned}$$

Hence, $D\omega(u, v)(p) = u^i(p)v^j(p)B_{i\,j}(p)$. Noting that $u_p = u^i(p)\partial_{i_{(p)}}$ and $v_p = v^j(p)\partial_{j_{(p)}}$ gives the desired result.

d) If $M = \mathbf{R}^2$, and d denotes the exterior derivative prove that $d\omega = T\omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\, dx \wedge dy$.

Solution: Let $w \in \Omega^1(\mathbf{R}^2)$ so we may write $\omega = f\,dx + g\,dy$ for some $f, g \in \mathbf{C}^\infty(\mathbf{R}^2)$. Then $d\omega = -\frac{\partial f}{\partial y} dx \wedge dy + \frac{\partial g}{\partial x} dx \wedge dy = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx \wedge dy$.

On the other hand, $T\omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{\partial}{\partial x}\omega(\frac{\partial}{\partial y}) - \frac{\partial}{\partial y}\omega(\frac{\partial}{\partial x})$, since $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$.

But $\frac{\partial}{\partial x}\omega(\frac{\partial}{\partial y}) = \frac{\partial g}{\partial x}$, and $\frac{\partial}{\partial y}\omega(\frac{\partial}{\partial x}) = \frac{\partial f}{\partial y}$, whence the result.

9. Let $M = \mathbf{S}^2 \setminus \{(x,0,z) \mid x \le 0\}$. In the usual coordinates φ, θ on M, define the **non-standard** tensor

$$h = -d\varphi \otimes d\varphi + 2\sin\varphi \; d\theta \otimes d\theta$$

a) Prove that h is a semi-Riemannian metric on M.

b) Find the non-zero Christoffel symbols for the Levi-Civita connection ∇ for h. You may assume that convenience that $\Gamma^k_{i\,j} = \frac{1}{2} h^{k\,k}(\partial_i h_{j\,k} - \partial_k h_{i\,j} + \partial_j h_{k\,i})$, in this case.

Solution:

Note that $[h_{i\,j}] = \begin{bmatrix} -1 & 0 \\ 0 & 2\sin\varphi \end{bmatrix}$ and $[h^{i\,j}] = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2\sin\varphi} \end{bmatrix}$. As for $\Gamma^{\varphi}_{-\,-}$, as both matrices are diagonal and do not depend on θ, only φ–derivatives contribute, and only in $\partial_\varphi h_{\theta\,\theta} = 2\cos\varphi$, so

$$\Gamma^{\varphi}_{\theta\,\theta} = \frac{1}{2} h^{\varphi\,\varphi}(-\partial_\varphi h_{\theta\,\theta}) = \frac{1}{2}(-1)(-2\cos\varphi) = \cos\varphi.$$

As for $\Gamma^{\theta}_{-\,-}$, we see that

$$\Gamma^{\theta}_{\theta\,\varphi} = \Gamma^{\theta}_{\varphi\,\theta} = \frac{1}{2} h^{\theta\,\theta}(\partial_\varphi h_{\theta\,\theta}) = \frac{1}{2}\frac{1}{2\sin\varphi}(2\cos\varphi) = \frac{\cot\varphi}{2}.$$

c) Show that the geodesic equations are

$$\ddot\theta + \dot\varphi\dot\theta\,\cot\varphi = 0 \quad \text{and} \quad \ddot\varphi + \dot\theta^2\,\cos\varphi = 0$$

Solution: $\ddot\gamma^i + \dot\gamma^j\dot\gamma^k\Gamma^i_{j\;k} = 0$ for '$i = \varphi$' is $\ddot\varphi + \dot\theta\dot\theta\;\Gamma^{\varphi}_{\theta\;\theta} = 0$, and

$$\ddot\varphi + \dot\theta\dot\theta\;\Gamma^{\varphi}_{\theta\;\theta} = 0 \iff \ddot\varphi + \dot\theta^2\,\cos\varphi = 0.$$

For

'$i = \theta$' is

$$\ddot\theta + \dot\varphi\;\dot\theta\;\Gamma^{\theta}_{\varphi\;\theta} + \dot\theta\;\dot\varphi\;\Gamma^{\theta}_{\theta\;\varphi} = 0 \iff \ddot\theta + \dot\varphi\dot\theta\,\cot\varphi = 0$$

d) Show that the first equation in (c) can be written as $\dfrac{d(\dot\theta\,\sin\varphi)}{dt} = 0.$

Solution: $\dfrac{d(\dot\theta\,\sin\varphi)}{dt} = \ddot\theta\sin\varphi + \dot\theta\dot\varphi\cos\varphi$; now divide by $\sin\varphi$ since it is never zero on M.

e) Find all <u>unit speed</u> geodesics $t \to \gamma(t)$ such that $t \to \varphi(\gamma(t))$ is constant.

Solution: If $\dot\varphi = 0$, then $\varphi = \varphi_0$ for some constant. Since $\dot\theta\,\sin\varphi = A$ is constant, $\dot\theta = \dfrac{A}{\sin\varphi_0}$. Unit speed is equivalent to $-\dot\varphi^2 + 2\sin\varphi\;\dot\theta^2 = 1 \iff \dot\theta^2 = \dfrac{A^2}{\sin^2\varphi_0} = \dfrac{1}{2\sin\varphi_0}$ yielding $A^2 = \dfrac{\sin\varphi_0}{2}$. However, $\dot\theta^2\cos\varphi_0 = 0$, and so as $\dot\theta \neq 0$, we find $\varphi_0 = \dfrac{\pi}{2}$ and hence $\varphi = \dfrac{\pi}{2}$ and $\theta = \pm\frac{\sqrt{2}}{2}t + B$ where B is any constant.

10. Suppose M is a smooth n–manifold and with connection D, and $\gamma : [a, b] \to M$ is a smooth curve in M.

a) If $v_1, \ldots, v_n$ are vector fields defined along γ, define "$\{v_1, \ldots, v_n\}$ *is a parallel frame along* γ."

b) Suppose $\{v_1, \ldots, v_n\}$ is a parallel frame along γ, and $v : [a, b] \to TM$ is any vector field along γ. If we write $v(t) = \xi^i(t)v_i(t)$ for $t \in [a, b]$ and smooth functions $\xi^i : [a, b] \to \mathbf{R}$, $i = 1, \ldots, n$, prove that

$$\left(D_{\dot\gamma(t)}v\right)(t) = \frac{d\xi^i}{dt}(t)v^i(t).$$

Solution:
Recall that if $w(t) = w^k(t)\partial_{k_{(\gamma(t))}}$, then

$$D_{\dot\gamma(t)}w(t) := \frac{dw^k}{dt}\partial_{k_{(\gamma(t))}} + \dot\gamma^i(t)w^j(t)A^k_{i\,j}(\gamma(t))\partial_{k_{(\gamma(t))}}.$$

Write $v_i = W^j_i\partial_{j_{(\gamma(t))}}$, so that $v(t) = \xi^i W^j_i \partial_{j_{(\gamma(t))}}$. Then,

$$\begin{aligned}
D_{\dot\gamma}v &= D_{\dot\gamma}(\xi^i W^j_i \partial_{j_{(\gamma(t))}}) \\
&= \frac{d}{dt}(\xi^i W^j_i)\partial_{j_{(\gamma(t))}} + \dot\gamma^l(t)\xi^i W^j_i A^k_{l\,j}(\gamma(t))\partial_{k_{(\gamma(t))}} \\
&= \frac{d}{dt}(\xi^i)v_i + \xi^i\frac{d}{dt}(W^j_i) + \dot\gamma^l(t)\xi^i W^j_i A^k_{l\,j}(\gamma(t))\partial_{k_{(\gamma(t))}} \\
&= \frac{d}{dt}(\xi^i)v_i + \xi^i D_{\dot\gamma}v_i \\
&= \frac{d}{dt}(\xi^i)v_i,
\end{aligned}$$

since $\{v_1, \ldots, v_n\}$ is a parallel frame along γ.

c) Let $M = \mathbf{S}^2 \setminus \{(x, 0, z) \mid x \le 0\}$ and $\gamma : [0, \frac{\pi}{2}) \to M$ be the smooth curve defined in (φ, θ) coordinates by

$$(\varphi(t), \theta(t)) = \left(\frac{\pi}{2} - t, 0\right).$$

That is, $(\varphi(t), \theta(t)) = (\varphi(\gamma(t)), \theta(\gamma(t))$.

Equip M with the Levi-Civita connection associated to the standard Riemannian metric inherited from $\mathbf{R}^3$. You may assume formula (E) on page 1 gives the only non-zero Christoffel symbols in these coordinates for this Levi-Civita connection.

Find a parallel frame along γ.

Solution: We know that $\dot\gamma(t) = -\partial_\varphi$, and from (E) that $\nabla_{\partial_\varphi}(\partial_\varphi) = 0$. Moreover $\nabla_{\partial_\varphi}(\partial_\theta) = \cot\varphi\,\partial_\theta$.Hence, since $\partial_\varphi \csc\csc\varphi = -\csc\varphi\cot\varphi$,

$$\begin{aligned}
\nabla_{\partial_\varphi}(\csc\varphi\,\partial_\theta) &= (\partial_\varphi\csc\varphi)\,\partial_\theta + \csc\varphi\nabla_{\partial_\varphi}(\partial_\theta) \\
&= -\csc\varphi\cot\varphi\,\partial_\theta + \csc\varphi\cot\varphi\,\partial_\theta \\
&= 0.
\end{aligned}$$

Since $\{\partial_\varphi, \csc\varphi\,\partial_\theta\}$ is linearly independent everywhere on M, the above shows that$\{\partial_{\varphi_{\gamma(t)}}, \csc\varphi(t)\,\partial_{\theta_{\gamma(t)}}\}$ is a parallel frame along γ.

11. Define C (an open 'cut' cone in $\mathbf{R}^3$) and H (an open subset of $\mathbf{R}^2$) as follows:

$$C = \Big\{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 = z^2,\ z > 0\Big\} \setminus \Big\{(x, 0, z) \mid x \leq 0\Big\}$$

$$H = \Big\{(r\cos\theta, r\sin\theta) \in \mathbf{R}^2 \mid r > 0, \theta \in (-\frac{\sqrt{2}\pi}{2}, \frac{\sqrt{2}\pi}{2})\Big\}.$$

Note that there are smooth global coordinates $r, \theta : H \to \mathbf{R}$ with

$$(x, y) = r(\cos\theta, \sin\theta), \quad \forall (x, y) \in H,$$

and there are smooth functions $\rho : C \to (0, \infty)$ and $\sigma : C \to (-\pi, \pi)$ satisfying

$$(x, y, z) = \rho(\cos\sigma, \sin\sigma, 1), \quad \forall (x, y, z) \in C.$$

A homeomorphism $\psi : H \to C$ is given by

$$\psi(p) = \frac{\sqrt{2}\, r}{2}(\cos[\sqrt{2}\theta)], \sin[\sqrt{2}\theta)], 1\,),$$

where $r = r(p)$ and $\theta = \theta(p)$. We use ψ to give C the differentiable structure that makes ψ a diffeomorphism. This makes the inclusion $C \xrightarrow{i} \mathbf{R}^3$ smooth.

Let $G = dx \otimes dx + dy \otimes dy + dz \otimes dz$ denote the standard metric on $\mathbf{R}^3$. Equip H with the metric $g = dr \otimes dr + r^2 d\theta \otimes d\theta$ and C with the metric $k = i^*G$. That is,

$$k = i^*dx \otimes i^*dx + i^*dy \otimes i^*dy + i^*dz \otimes i^*dz.$$

a) Show that on C, $k = 2\, d\rho \otimes d\rho + \rho^2 d\sigma \otimes d\sigma$.

Solution: Noting that $i^*x = \rho\cos\sigma, i^*y = \rho\sin\sigma$, and $i^*z = \rho$,

$$\begin{aligned} i^*dx \otimes i^*dx &= d(i^*x) \otimes d(i^*x) \\ &= d(\rho\cos\sigma) \otimes d(\rho\cos\sigma) \\ &= (\cos\sigma d\rho - \rho\sin\sigma d\sigma) \otimes (\cos\sigma d\rho - \rho\sin\sigma d\sigma) \\ &= \cos^2\sigma d\rho \otimes d\rho + \rho^2\sin^2\sigma d\sigma \otimes d\sigma - \rho\sin\sigma\cos\sigma(d\rho \otimes d\sigma + d\sigma \otimes d\rho), \end{aligned}$$

while

$$\begin{aligned} i^*dy \otimes i^*dy &= d(i^*y) \otimes d(i^*y) \\ &= d(\rho\sin\sigma) \otimes d(\rho\sin\sigma) \\ &= (\sin\sigma d\rho + \rho\cos\sigma d\sigma) \otimes (\sin\sigma d\rho + \rho\cos\sigma d\sigma) \\ &= \sin^2\sigma d\rho \otimes d\rho + \rho^2\cos^2\sigma d\sigma \otimes d\sigma + \rho\sin\sigma\cos\sigma(d\rho \otimes d\sigma + d\sigma \otimes d\rho), \end{aligned}$$

and $i^*dz \otimes i^*dz = d(i^*z) \otimes d(i^*z) = d\rho \otimes d\rho$. Hence, $k = 2\, d\rho \otimes d\rho + \rho^2 d\sigma \otimes d\sigma$

b) Show that $\forall p \in H$, $\rho \circ \psi(p) = \dfrac{\sqrt{2}\, r(p)}{2}$, and $\sigma \circ \psi(p) = \sqrt{2}\, \theta(p)$.

Solution: Since $\psi(p) = \frac{\sqrt{2}\, r}{2}(\cos[\sqrt{2}\theta)], \sin[\sqrt{2}\theta)], 1\,) = \rho(\cos\sigma, \sin\sigma, 1)$, equating the last compnents gives $\rho \circ \psi = \dfrac{\sqrt{2}\, r}{2}$. Then, equality of the first two components (noting the ranges of σ and θ) forces $\sigma \circ \psi = \sqrt{2}\, \theta$.

c) If we define $\psi^*(f\alpha \otimes \beta) = \psi^*(f)\psi^*(\alpha) \otimes \psi^*(\beta)$ for $\alpha, \beta \in \Omega^1(C)$, and $f \in \mathbf{C}^\infty(C)$, and extend by linearity, show that

$$\psi^*(k) = g$$

Solution: This follows immediately from (a) and (b), upon noting that (b) says $\psi^*\rho = \dfrac{\sqrt{2}\, r}{2}$ and $\psi^*\sigma = \sqrt{2}\theta$.

12. Suppose (M, g) is a semi-Riemannian manifold and with Levi-Civita connection ∇, and let R denote the Riemann curvature defined as usual by

$$\mathrm{R}(u,v)w = (\nabla_u \nabla_v - \nabla_v \nabla_u)w - \nabla_{[u,v]}w, \quad \forall u, v, w \in \mathrm{Vect}(M).$$

a) Prove that $\mathrm{R}(-,-)-$ is $\mathbf{C}^\infty(M)$-linear in the third slot.

b) Suppose that u, v, w are commuting vector fields. Prove that

$$\mathrm{R}(u,v)w + \mathrm{R}(w,u)v + \mathrm{R}(v,w)u = 0.$$

(Hint: Use the fact that the Levi-Civita connection ∇ is torsion-free to show that e.g., $(\nabla_u \nabla_v - \nabla_v \nabla_u)w = \nabla_u \nabla_v w - \nabla_v \nabla_w u$, in order to cancel terms in pairs.)

c) Suppose that u, v, w, x and z are commuting vector fields. Expand $[u,v](g(w,x))$, (which is zero since $[u,v] = 0$), using the fact that ∇ is a metric connection to show that

$$g(\mathrm{R}(u,v)w, x) + g(w, \mathrm{R}(u,v)x) = 0.$$

Solution: These are all straighforward computations.

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13. Suppose (M, g) is a semi-Riemannian manifold with Levi-Civita connection ∇, and let R denote its Riemann curvature tensor. Define a tensor field of type (0,2) on M, the *Ricci* tensor by

$$\mathrm{Ric}(u,v) = \text{the trace of the linear map } \Big[w \mapsto \mathrm{R}(u,w)v\Big], \quad \forall u, v, w \in \mathrm{Vect}(M)$$

(Note the placement of w in $\mathrm{R}(u,w)v$.)

a) Show that $\forall u, v \in \mathrm{Vect}(M)$

$$\mathrm{tr}\Big[w \mapsto \mathrm{R}(u,v)w\Big] = 0.$$

(Note the placement of w in $\mathrm{R}(u,v)w$. This not the Ricci tensor. You may use Q.12.)

Solution: Fix u, v and denote by T the map $w \mapsto \mathrm{R}(u,v)w$. Suppose $g(e_i, f_j) = \delta_{i\,j}$ We know that

$$2\,\mathrm{tr}\,T = \sum_i g(Te_i, f_i) + \sum_i g(e_i, Tf_i) = \sum_i \Big(g(\mathrm{R}(u,v)e_i, f_i) + g(e_i, \mathrm{R}(u,v)f_i)\Big) = 0$$

by the previous question.

b) Show that $\mathrm{Ric}(u,v) = \mathrm{Ric}(v,u), \forall u, v \in \mathrm{Vect}(M)$. (You may use Q. 1(d) and Q. 12 without proof.)

Solution: Fix $u, v \in \mathrm{Vect}(M)$, and define $S(w) = R(u,w)v$, $T(w) = R(u,v)w$ and $U(w) = R(v,w)u$. Then 12(b) is equivalent to

$$T - S + U = 0.$$

Since the trace is linear $\mathrm{tr}\,T - \mathrm{tr}\,S + \mathrm{tr}\,U = 0$. But by (a), $\mathrm{tr}\,T = 0$. Hence $\mathrm{tr}\,S = \mathrm{tr}\,U$, i.e. $\mathrm{Ric}(u,v) = \mathrm{Ric}(v,u)$.

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14. Suppose (M, g) and (N, h) are semi-Riemannian manifolds with Levi-Civita connections ∇ and ∇' respectively. Suppose that $s : M \to N$ is a diffeomorphism which is also an isometry of semi-Riemannian manifolds, that is, $s^*h = g$. Explicitly, this means that

$$g(u, v) = h(s_*u, s_*v) \circ s, \quad \forall u, v \in \mathrm{Vect}(M)$$

a) Use formula (C) on the first page to prove that

$$s_*(\nabla_u v) = \nabla'_{s_*u} s_*v, \quad \forall u, v \in \mathrm{Vect}(M)$$

(You may assume when using formula (C) that all vector fields commute.)

b) If R and R$'$ respectively denote the Riemann curvature tensors for (M, g) and (N, h), show that

$$s_*(\mathrm{R}(u, v)w) = R'(s_*u, s_*v)s_*w, \quad \forall u, v, w \in \mathrm{Vect}(M).$$

(You may again assume (w.l.o.g.) that all vector fields commute.)

c) [Bonus] Use (b) to compute the Riemann curvature tensor for Levi-Civita connection on the Riemannian manifold (C, k) of question 11.

Solution: (a) Let $u, v, w \in \mathrm{Vect}(M)$. Then by formula (C),

$$2h(\nabla'_{s_*u}, s_*w) = s_*(u)(h(s_*v, s_*w)) - s_*(w)(h(s_*u, s_*v)) + s_*(v)(h(s_*w, s_*u)).$$

But if $f \in \mathbf{C}^\infty(N)$, $(s_*u)(f) = [u(f \circ s)] \circ s^{-1}$, so

$$\begin{aligned} 2h(\nabla'_{s_*u}, s_*w) \circ s &= u(h(s_*v, s_*w) \circ s) \circ s^{-1} \circ s - w(h(s_*u, s_*v) \circ s) \circ s^{-1} \circ s + v(h(s_*w, s_*u) \circ s) \circ s^{-1} \circ s \\ &= u\, g(v, w) - w\,(g(u, v)) + v\, g(w, u) \\ &= 2g(\nabla_u v, w) \\ &= 2h(s_*(\nabla_u v), s_*w) \circ s. \end{aligned}$$

Since s_* is an isomorphism, this holds for all w,and h is non-degenerate, $s_*(\nabla_u v) = \nabla'_{s_*u} s_*v, \quad \forall u, v \in \mathrm{Vect}(M)$.

b) Well, $s_*(\nabla_u \nabla_v w) = \nabla'_{s_*u} s_*(\nabla_v w) = \nabla'_{s_*u} \nabla'_{s_*v} s_*w$. So

$$s_*R(u, v)w = [\nabla'_{s_*u}, \nabla'_{s_*v}]s_*w = R'(s_*u, s_*v)s_*w$$

c) We know from (b), since ψ of Q. 11 is an isometry, and (using the notation of Q.11) $\psi^*(g) = k$, if the Riemann curvature of either g or k is zero, both will be. But the metric g of Q.11 is the standard flat metric on the plane (pulled back to the subset H). So the Riemann curvature of k is also zero. (The cone is Riemman-'flat'!)