# Faculty of Science 》 <br> Department of <br> mathematics and statistics 



## MAT 4183 Mid-term Exam

4 November, 2015. Duration: 80 minutes

Family Name: $\qquad$

First Name: $\qquad$

Student number: $\qquad$

| $[9] 1$ |  |
| :---: | :--- |
| $[7] 2$ |  |
| $[6] \mathbf{3}$ |  |
| $[9] \mathbf{4}$ |  |
| $[8] \mathbf{5}$ |  |
| $[7] 6$ |  |
| $[9] 7$ |  |
| $[7] 8$ |  |
| Total |  |

1. You have 80 minutes to complete this exam. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All implanted cyber devices not necessary for life-support must be disabled at the beginning of the exam.
2. The correct answer requires reasonable justification written legibly and logically.
3. You must attempt at least two of questions 3,4 and 5 , which will count in your total. However, 30 points will earn $\mathbf{1 0 0 \%}$. The marks for each question are indicated above and inside the paper. (It is possible to score more than $100 \%$.)
4. Unless otherwise stated, all vector spaces are over the reals and may be assumed to be finite dimensional.
5. Please use the space provided, including the backs of pages if necessary. If you need scrap paper, please ask.
6. Good luck! Bonne chance!
7. [Total: 9] Let $V W, X$ and $Y$ be finite dimensional real vector spaces. Suppose $S: V \rightarrow V$ and $T: W \rightarrow W$ are linear maps. Define a bilinear map $S \times T: V \times W \rightarrow V \otimes W$ by

$$
(S \times T)(v, w)=S(v) \otimes T(w), \quad \text { for }(v, w) \in V \times W
$$

Recall the natural isomorphisms $Y \otimes X^{*} \xrightarrow{e} \operatorname{Hom}(X, Y)$, and $V^{*} \otimes W^{*} \xrightarrow{t}(V \otimes W)^{*}$.
Let $\left\{v_{i}\right\}_{i=1}^{n}$ be a basis for $X$, and $\left\{v^{i}\right\}_{i=1}^{n}$ the dual basis. Recall that the trace

$$
\operatorname{tr}: \operatorname{Hom}(X, X) \rightarrow \mathbf{R}
$$

satisfies $\operatorname{tr}(T)=v^{i}\left(T\left(v_{i}\right)\right), \forall T \in \operatorname{Hom}(X, X)$.
a) [2] Give the formulas for the natural isomorphisms $Y \otimes X^{*} \xrightarrow{e} \operatorname{Hom}(X, Y)$ and $V^{*} \otimes W^{*} \xrightarrow{t}$ $(V \otimes W)^{*}$ on elements of rank one in $Y \otimes X^{*}$ and $V^{*} \otimes W^{*}$ respectively.
b) [1] Explain briefly why there is a unique linear map $S \otimes T: V \otimes W \rightarrow V \otimes W$ such that

$$
(S \otimes T)(v \otimes w)=S(v) \otimes T(w), \quad \text { for } v \in V, w \in W
$$

c) [2] Suppose $S=e(v \otimes f)$ for $v \in V, f \in V^{*}$. Show that $\operatorname{tr}(S)=f(v)$.
d) [2] Suppose $S=e(v \otimes f)$ for $v \in V, f \in V^{*}$ and $T=e(w \otimes g)$ for $w \in W, g \in W^{*}$. Show that $e((v \otimes w) \otimes t(f \otimes g))=S \otimes T$. (Hint: Note that is suffices to check equality on rank one elements of $V \otimes W$.)
e) [2] Prove that $\operatorname{tr}(S \otimes T)=\operatorname{tr}(S) \operatorname{tr}(T)$, for all $S \in \operatorname{Hom}(V, V)$ and $T \in \operatorname{Hom}(W, W)$.
(Hint: Note that both sides are bilinear in $S$ and $T$, so it suffices to check equality in the case where $S=e(v \otimes f)$ and $T=e(w \otimes g)$ for $v \in V, f \in V^{*}, w \in W, g \in W^{*}$.)
2. [Total: 7] Let $A=\left[\begin{array}{ccc}0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0\end{array}\right]$ and define $T \in \operatorname{Hom}\left(\mathbf{R}^{3}, \mathbf{R}^{3}\right)$ by $T(v)=A v$.

Recall the isomorphism $e: \mathbf{R}^{3} \otimes\left(\mathbf{R}^{3}\right)^{*} \rightarrow \operatorname{Hom}\left(\mathbf{R}^{3}, \mathbf{R}^{3}\right)$ satisfying

$$
e(v \otimes f)(w)=f(w) v
$$

and let $t=e^{-1}(T)$.
a) [2] Find an explicit expression for $t \in \mathbf{R}^{3} \otimes\left(\mathbf{R}^{3}\right)^{*}$.
b) [3] Write $t=\sum_{i=1}^{2} v_{i} \otimes w^{i}$ for $v_{i} \in \mathbf{R}^{3}, w^{i} \in\left(\mathbf{R}^{3}\right)^{*}$.
c) [2] Use (b) to find ordered bases $\mathcal{A}=\left\{u_{1}, u_{2}, u_{3}\right\}$ and $\mathcal{B}=\left\{x_{1}, x_{2}, x_{3}\right\}$ of $\mathbf{R}^{3}$ such that the matrix of $T$ w.r.t. $\mathcal{A}$ and $\mathcal{B}$ is

$$
\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

3. [Total: 6] Let $M$ be a smooth $n$-dimensional manifold with maximal atlas $\mathcal{A}$.

Let $U \xrightarrow{j} M$ denote the inclusion of an open subset in $M$. If $V$ is an open subset of $\mathbf{R}^{n}$, give $V$ the smooth structure inherited from the standard smooth structure on $\mathbf{R}^{n}$.
a) [2] Define what it means for $(U, \varphi)$ to be a local coordinate system for $M$.

Now Suppose $(U, \varphi)$ is a local coordinate system for $M$, and define

$$
\mathcal{B}=\{(U, \varphi)\} .
$$

b) [2] Prove that $\mathcal{B}$ is a smooth atlas for $U$, and that with this smooth structure on $U$, the inclusion $U \xrightarrow{j} M$ is a smooth map.
c) [2] Prove that $\varphi: U \rightarrow \varphi(U)$ is an diffeomorphism.
4. [Total: 9] Suppose $M$ and $N$ are smooth manifolds, that $p \in M$, and $M \xrightarrow{\phi} N$ is smooth.
a) [2] Define what is meant by a tangent vector $v_{p}$ at $p$.
b) [3] Show that if $\gamma: \mathbf{R} \rightarrow M$ is a smooth curve, that the map $\gamma^{\prime}\left(t_{0}\right): \mathbf{C}^{\infty}(M) \rightarrow \mathbf{R}$ defined by

$$
\gamma^{\prime}\left(t_{0}\right)(f)=\frac{d(f(\gamma(t))}{d t}\left(t_{0}\right), \quad \forall f \in \mathbf{C}^{\infty}(M)
$$

is a tangent vector at $p=\gamma\left(t_{0}\right)$.
c) [2] Define the tangent map of $\phi$,

$$
\phi_{*}: T_{p} M \rightarrow T_{\phi(p)} N,
$$

and show carefully that $\phi_{*}$ really does map into $T_{\phi(p)} N$.
d) [2] Let $M=N=\mathbf{R}^{3}, A \in \mathbf{M}_{33}(\mathbf{R})$ be a fixed 3 by 3 matrix, and define $\mathbf{R}^{3} \xrightarrow{\phi} \mathbf{R}^{3}$ by $\phi(u)=A u$. If $p \in \mathbf{R}^{3}$ and $v_{p}=a \frac{\partial}{\partial x}(p)+b \frac{\partial}{\partial y}(p)+b \frac{\partial}{\partial z}(p) \in T_{p} \mathbf{R}^{3}$, compute $\phi_{*}\left(v_{p}\right)$
5. [Total: 8] Let $M=\mathbf{R}^{2} \backslash\{(0,0)\}$ and define $\omega \in \Omega^{1}(M)$ and $v \in \operatorname{Vect}(M)$ by

$$
\omega=-y d x+x d y,
$$

and

$$
v=y \frac{\partial}{\partial x}-x \frac{\partial}{\partial y} .
$$

a) [1] Define an integral curve of a vector field $u \in \operatorname{Vect}(M)$.
b) [2] Define the flow $\phi_{t}: M \rightarrow M$ generated by an integrable vector field $u \in \operatorname{Vect}(M)$.
c) [2] Show that $v$ is integrable, and find a formula for the flow $\phi_{t}: M \rightarrow M$.
d) [3] Is $\omega$ invariant by $\phi_{t}$ ?
6. [Total: 7] Assume that the stereographic projections

$$
\begin{aligned}
p_{N} & : \mathbf{S}^{1} \backslash\{(0,1)\} \rightarrow \mathbf{R} \\
p_{S} & : \mathbf{S}^{1} \backslash\{(0,-1)\} \rightarrow \mathbf{R}
\end{aligned}
$$

can be assembled to form a smooth atlas for $\mathbf{S}^{1}$, and equip $\mathbf{S}^{1}$ with this smooth structure. Let $i: \mathbf{S}^{1} \rightarrow \mathbf{R}^{2}$ denote the inclusion.
a) [3] Prove that $i_{*}: T_{p} \mathbf{S}^{1} \rightarrow T_{p} \mathbf{R}^{2}$ is injective for all $p \in \mathbf{S}^{1}$.
(You may assume that $\forall f \in \mathbf{C}^{\infty}\left(\mathbf{S}^{1}\right), \exists h \in \mathbf{C}^{\infty}\left(\mathbf{R}^{2}\right)$ such that $\forall z \in \mathbf{S}^{1}, h(z)=f(z)$.)
b) [4] Let $U=\left\{p \in \mathbf{S}^{1} \mid x(p)>0\right\}$ and $\theta: U \rightarrow \mathbf{R}$ be defined by $\theta(p)=\arctan \left(\frac{y(p)}{x(p)}\right)$. You may assume that $(U, \theta)$ is a local coordinate system for $\mathbf{S}^{1}$ and that if $\sigma=\theta^{-1}$, then $\sigma(t)=$ $(\cos t, \sin t), \forall t \in \theta(U)$.

Show carefully that if $\frac{\partial}{\partial \theta}:=\sigma_{*}\left(\frac{d}{d t}\right)$, and $p=(a, b) \in \mathbf{S}^{1}$, then

$$
i_{*}\left(\frac{\partial}{\partial \theta}(p)\right)=-b \frac{\partial}{\partial x}(p)+a \frac{\partial}{\partial y}(p) .
$$

7. [Total: 9] Let $M$ be an open subset of $\mathbf{R}^{n}$, and $\omega=\omega_{i} d x^{i} \in \Omega^{1}(M)$ be a 1-form on $M$, and $\beta:[a, b] \rightarrow U$ a smooth curve (since $[a, b]$ isn't open in $\mathbf{R}$, this means the usual thing.)

We define $\int_{\beta} \omega$, the integral of $\omega$ over $\beta$ by

$$
\int_{\beta} \omega=\int_{a}^{b} \omega\left(\beta^{\prime}(t)\right) d t
$$

Note that if $\beta(t)=\beta^{j}(t) e_{j}$, so that $\beta^{\prime}(t)=\frac{d \beta^{j}(t)}{d t} \frac{\partial}{\partial x^{j}}{ }^{\beta(t)}$, then

$$
\omega\left(\beta^{\prime}(t)\right)=\omega_{i}(\beta(t)) d x^{i}\left(\beta^{\prime}(t)\right)=\omega_{i}(\beta(t)) d x^{i}\left(\frac{d \beta^{j}}{d t} \frac{\partial}{\partial x^{j}}{ }^{\beta(t)}\right)=\omega_{i}(\beta(t)) \frac{d \beta^{i}}{d t} .
$$

Now let $M=\mathbf{R}^{2} \backslash\{0\}$, define a curve $\beta$ in $M$ by $\beta(t)=\left[\begin{array}{c}\cos t \\ \sin t\end{array}\right]$ for $t \in[0,2 \pi]$, and $\omega \in \Omega^{1}(M)$ by

$$
\omega=-\frac{y}{x^{2}+y^{2}} d x+\frac{x}{x^{2}+y^{2}} d y
$$

a) [1] Show that $\omega\left(\beta^{\prime}(t)\right)=1, \quad \forall t \in[0,2 \pi]$.
b) [1] Show that $\int_{\beta} \omega=2 \pi$.
c) [2] Show that if $\eta \in \Omega^{1}(M)$ and $\eta=d f$ for some smooth function $f \in \mathbf{C}^{\infty}(M)$, then $\int_{\beta} \eta=0$. (We may thus conclude that $\omega \neq d f$ for any smooth function $\theta \in \mathbf{C}^{\infty}(M)$.)
d) [3] Now let $N=\mathbf{R}^{2} \backslash\{(x, y) \mid x \leq 0\}$, and define $\omega \in \Omega^{1}(N)$ exactly as in (a) above. Show that there is a smooth function $f \in \mathbf{C}^{\infty}(N)$ such that $\omega=d f$.
e) [2] Use the definition given in class for the exterior derivative in $\mathbf{R}^{n}$ to show that $d \omega=0$.
8. [Total: 7] Let $M$ be a smooth $n$-manifold.
a) [1] Define tangent bundle $T M$ of $M$. (You do not need to give an atlas for $T M$.)
b) [4] Prove that there is a diffeomorphism $\psi: T \mathbf{S}^{1} \rightarrow S^{1} \times \mathbf{R}$ which makes the diagram

commutative iff there is $v \in \operatorname{Vect}\left(\mathbf{S}^{1}\right)$ such that $v_{p} \neq 0, \forall p \in \mathbf{S}^{1}$.
c) [2] Prove that there is $v \in \operatorname{Vect}\left(\mathbf{S}^{1}\right)$ such that $v_{p} \neq 0, \forall p \in \mathbf{S}^{1}$. (Hint: consider the tangent to the curve $\beta$ defined in the previous question.)

