2. [Total: 7] Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ and define $T \in \operatorname{Hom}(\mathbf{R}^3, \mathbf{R}^3)$ by $T(v) = Av$.

Recall the isomorphism $e : \mathbf{R}^3 \otimes (\mathbf{R}^3)^* \to \operatorname{Hom}(\mathbf{R}^3, \mathbf{R}^3)$ satisfying

$$e(v \otimes f)(w) = f(w)v,$$

and let $t = e^{-1}(T)$.

a) [2] Find an explicit expression for $t \in \mathbf{R}^3 \otimes (\mathbf{R}^3)^*$.

Solution: Write $A = [\, v_1 \quad v_2 \quad v_3 \,]$ in block column form, so we know that $T(e_i) = v_i$, $(i = 1, 2, 3)$ where $\{e_1, e_2, e_3\}$ is the standard ordered basis of $\mathbf{R}^3$. Now, set

$$t = v_1 \otimes e^1 + v_2 \otimes e^2 + v_3 \otimes e^3,$$

where $\{e^1, e^2, e^3\}$ is the basis of $(\mathbf{R}^3)^*$ dual to $\{e_1, e_2, e_3\}$. Then it's clear from the defintion of e that $e(t) = T$.

b) [3] Write $t = \sum_{i=1}^{2} v_i \otimes w^i$ for $v_i \in \mathbf{R}^3, w^i \in (\mathbf{R}^3)^*$.

Solution: Noting that $v_2 = v_1 + v_3$, we see that $t = v_1 \otimes (e^1 + e^2) + v_3 \otimes (e^2 + e^3)$.

c) [2] Use (b) to find ordered bases $\mathcal{A} = \{u_1, u_2, u_3\}$ and $\mathcal{B} = \{x_1, x_2, x_3\}$ of $\mathbf{R}^3$ such that the matrix of T w.r.t. $\mathcal{A}$ and $\mathcal{B}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: Recall from (b) that $t = v_1 \otimes (e^1 + e^2) + v_3 \otimes (e^2 + e^3)$ and so $\{v_1, v_3\}$ is a basis for $\operatorname{im} T$ and the fact that $v_2 = v_1 + v_3$ also means that $v = (1, -1, 1) \in \ker T$ (so we'll take $u_3 = v$ in a moment).

So set $x_1 = v_1, x_2 = v_3$ and $x_3 = e_2$ (the latter has many choices).

Now (using (b) again) it remains to find two vectors u_1, u_2 with $(e^1+e^2)(u_1) = 1$, $(e^1+e^2)(u_2) = 0$, $(e^2 + e^3)(u_1) = 0$, $(e^2 + e^3)(u_2) = 1$.

It 's easy to see that $u_1 = e_1$ and $u_2 = e_3$ satisfy these 4 conditions, and we conclude by setting $u_3 = v$. Then $T(u_i) = x_i$ for $i = 1, 2$, and $T(u_3) = 0$, as required.