Fall 2016 MAT 1341A Test 4 (V.2) Solutions

1. Let A be an 18×16 matrix such that $Ax = 0$ has only the trivial solution. Answer the following questions:

$\therefore$ rank $A = 16$

- What is the rank of A?
- If $Ax = b$ is consistent for some $b \in \mathbf{R}^{18}$, will it have a unique solution?

A. 0, Yes
B. 16, Yes ✓
C. 16, No
D. 18, Yes
E. 18, No
F. 2, Yes

$$\begin{array}{c|c|} & 16 \\ 18 & \boxed{A} \end{array}$$

If rank $A = 16$ and $Ax = b$ is consistent, $[A|b] \sim \left[\begin{array}{c|c} I_{16} & * \\ 0 \cdots 0 & 0 \\ 0 \cdots 0 & 0 \end{array}\right]$

There will be no parameters in the gen'l solution to $Ax=b$ $\therefore$ sol^n is unique

2. Let $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$. The third row of B^{-1} is:

A. $[2 \ -1 \ -1\,]$
B. $[-1 \ 1 \ 1\,]$
C. $[-3 \ 1 \ 2\,]$
D. $[0 \ 0 \ 1\,]$
E. $[1 \ 1 \ -1\,]$
F. B is not invertible.

$$[B|I_3] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right]$$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 2 \\ 2 & 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (*)$$

a) Find a basis for $\ker A = \{x \in \mathbf{R}^4 \mid Ax = 0\}$.

b) Find a basis for the column space of A.

c) Find a basis for the row space of A.

d) Find the dimension of $\{Ax \mid x \in \mathbf{R}^4\}$.

a) $[A|0] \underset{\text{above}}{\overset{\text{from}}{\sim}} \begin{array}{cccc} & s & & t \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\therefore$ $x_1 = -s+t$, $x_2 = s$, $x_3 = 3t$, $x_4 = t$; $s,t \in \mathbb{R}$

$\therefore \{(-1,1,0,0),(1,0,3,1)\}$ is a basis of $\ker A$ (b/c it is the set of basic sol'ns).

b) Since the leading ones in the RRE form of A sit in columns 1 & 3, $\{(1,1,2),(0,-1,-1)\}$ is a basis for col A. (Because of the col. space alg.)

c) By $(*)$ and the row space algorithm,
$\{(1,1,0,-1),(0,0,1,-3)\}$ is a basis for row(A)

d) We note that $\{Ax \mid x \in \mathbb{R}^4\} = \text{col}(A)$, which has dimension equal to rank $A = 2$ (OR, see part (b).)

4. Let $v_1 = (0,1,0,1), v_2 = (0,0,1,0), v_3 = (0,1,0,-1)$, and $W = \text{span}\{v_1, v_2, v_3\}$.

a) Show that $\{v_1, v_2, v_3\}$ is an orthogonal set.

b) Briefly explain why $\{v_1, v_2, v_3\}$ is a basis of W

c) Find the best approximation to (1, 1, 2, −1) by vectors in W.

d) Extend $\{v_1, v_2, v_3\}$ to a basis of $\mathbf{R}^4$.

a) $v_1 \cdot v_2 = 0$; $v_1 \cdot v_3 = 1-1 = 0$, $v_2 \cdot v_3 = 0$; moreover $v_1 \neq 0$, $v_2 \neq 0$ and $v_3 \neq 0$, so $\{v_1, v_2, v_3\}$ is orthogonal.

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b) Since (by definition of W), $\{v_1, v_2, v_3\}$ spans W, and by (a) $\{v_1, v_2, v_3\}$ is orthog. and so l.i., $\{v_1, v_2, v_3\}$ is a basis of W

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c) We compute

$$\text{proj}_W(1,1,2,-1) = \left(\frac{(1,1,2,-1)\cdot v_1}{\|v_1\|^2}\right) v_1 + \left(\frac{(1,1,2,-1)\cdot v_2}{\|v_2\|^2}\right) v_2 + \left(\frac{(1,1,2,-1)\cdot v_3}{\|v_3\|^2}\right) v_3$$

$$= 0 v_1 + \frac{2}{1}(0,0,1,0) + \frac{2}{2}(0,1,0,-1) = \underline{(0,1,2,-1)}$$

$\sqrt{//}$

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d) Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ & v_4 & & \end{bmatrix}$. We will choose v_4 so that rank $A = 4$.

Well, $A \sim \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ & v_4 & & \end{bmatrix}$, so we set

$v_4 = (1,0,0,0)$. Then, $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$ that contains $\{v_1, v_2, v_3\}$.

5(a). State whether the following is true or false, and justify your answer.

- If you say the statement may be false, you must give an explicit example - with numbers, matrices, or functions (as is appropriate), if possible, or an argument using theorems and facts from class.

- If you say the statement is always true, you must give a clear explanation.

(i) Let $\{u, v, w, x\}$ be a basis for $\mathbf{R}^4$. If A is an invertible 4×4 matrix, then $\{Au, Av, Aw, Ax\}$ is also a basis for $\mathbf{R}^4$.

Since $\dim \mathbb{R}^4 = 4$, it suffices to show that $\{Au, Av, Aw, Ax\}$ is l.i.: Suppose $aAu + bAv + cAw + dAx = 0$ for scalars a, b, c, d. Since A is invertible, multiplying both sides of this eqn by A^{-1} (on the left) yields $au + bv + cw + dx = 0$. But $\{u, v, w, x\}$ is l.i., so $a = b = c = d = 0$. Hence $\{Au, Av, Aw, Ax\}$ is l.i. & hence a basis of $\mathbb{R}^4$

ANSWER $\boxed{\text{TRUE}}$

(ii) The columns of a 5 x 3 matrix are always linearly dependent.

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Clearly, rank $A = 3$, so the columns of A are l.i.

ANSWER $\boxed{\text{FALSE}}$

5(b). Let A be a $n \times n$ matrix with real entries. Give three additional statements **equivalent to**

"There is $x \in \mathbf{R}^n$ with $x \neq 0$ such that $Ax = 0$,"

one each in terms of:

(i) The *inhomogeneous* system $Ax = b$ (i.e. where $b \neq 0$):

This may be inconsistent. (Since A is not invertible.)

(ii) The row space of A: has dimension $< n$.

(iii) the invertibility (or not) of A^t: Since A is not invertible, neither is A^t

(The above are all consequences of the Invertible Matrix Theorem.)

6. [Bonus/Challenge]

Suppose A is an invertible 15×15 matrix and B is any 15×12 matrix with $\operatorname{rank} B = 12$. Prove carefully that $\operatorname{rank} AB = 12$.

(You <u>cannot</u> choose A or B: your proof must work for all A and B satisfying the conditions above.)

Let $B = [b_1 \ldots b_{12}]$ be B in block column form.

We know (since $\operatorname{rank} B = 12$) that $\{b_1, \ldots, b_{12}\}$ is l.i.

Now $AB = [Ab_1 \; Ab_2 \; \ldots \; Ab_{12}]$ (ie $Ab_j = j^{th}$ col. of AB)

It suffices to show that $\{Ab_1, \ldots, Ab_{12}\}$ is l.i., since $\operatorname{rank} AB \leq 12$.

Suppose $a_1 Ab_1 + \cdots + a_{12} Ab_{12} = 0$ for scalars $a_1, \ldots, a_{12}$.

Multiplying by A^{-1} on the left of both sides of this equation, we obtain $a_1 b_1 + \cdots + a_{12} b_{12} = 0$.

But $\{b_1, \ldots, b_{12}\}$ is l.i., so $a_1 = a_2 = \cdots = a_{12} = 0$.

Hence $\{Ab_1, \ldots, Ab_{12}\}$ is l.i. and thus

$\operatorname{rank} AB = 12$.