Mat 1341A F16 Test 4 (V1) Solutions

1. Let A be an 8×6 matrix such that $Ax = 0$ has only the trivial solution. Answer the following questions:

$8 \overset{6}{\boxed{A}}$ $\therefore$ rank $A = 6$

- What is the rank of A?
- Is $Ax = b$ consistent for all $b \in \mathbf{R}^8$?

A. 0, Yes
B. 6, Yes
C. 6, No
D. 8, Yes
E. 8, No
F. 2, Yes

Since A is 6×8, and rank $A = 6$, dim col $A = 6 < 8$. Hence there are vectors b in $\mathbb{R}^8$ s.t. $Ax = b$ is not consistent. (Recall: $Ax = b$ is consistent $\Leftrightarrow$ $b \in$ col A.)

2. Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. The third row of B^{-1} is:

A. [1 2 1]
B. [−1 1 1]
C. [−3 1 2]
D. [0 0 1]
E. [1 1 −1]
F. B is not invertible.

$$[B \mid I_3] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right]$$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad *$$

a) Find a basis for the row space of A.

b) Find a basis for the column space of A.

c) Find a basis for $\ker A = \{x \in \mathbf{R}^4 \mid Ax = 0\}$.

d) Find the dimension of $\{Ax \mid x \in \mathbf{R}^4\}$.

c) $[A|0] \overset{\text{from}}{\underset{\text{above}}{\sim}} \begin{array}{cccc} & s & & t \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \therefore \begin{array}{l} x_1 = -s-t \\ x_2 = s \\ x_3 = 3s \\ x_4 = t \end{array} \; ; s,t \in \mathbb{R}$

$\therefore \{(-1,1,0,0), (1,0,3,1)\}$ is a basis for ker A

(Since it consists of the basic solutions.)

b) Since the leading ones in the RRE form of A sit in columns 1 & 3, $\{(1,2,1), (0,-1,-1)\}$ is a basis for col A.
(By the col. space alg.)

a) By (*) and the row space alg.,
$\{(1,1,0,-1), (0,1,1,-3)\}$ is a basis for row(A)

d) Since $\{Ax \mid x \in \mathbb{R}^4\} = \text{col}(A)$, & dim col A = rank A = 2,
$\dim\{Ax \mid x \in \mathbb{R}^4\} = 2$.

4. Let $u_1 = (0,1,1,0), u_2 = (0,0,0,1), u_3 = (0,1,-1,0)$, and $U = \text{span}\{u_1, u_2, u_3\}$.

a) Show that $\{u_1, u_2, u_3\}$ is an orthogonal set.

b) Briefly explain why $\{u_1, u_2, u_3\}$ is a basis of U.

c) Find the best approximation to $(1,\ -1,\ 2,\ -1)$ by vectors in U.

d) Extend $\{u_1, u_2, u_3\}$ to a basis of $\mathbf{R}^4$.

a) $u_1 \cdot u_2 = 0$, $u_1 \cdot u_3 = 1 - 1 = 0$, $u_2 \cdot u_3 = 0$. Moreover, each of u_1, u_2 & u_3 is non-zero. Hence $\{u_1, u_2, u_3\}$ is orthogonal.

---

b) Since $\{u_1, u_2, u_3\}$ spans U (by defn of U), and (a) shows $\{u_1, u_2, u_3\}$ is orthog. and so l.i., $\{u_1, u_2, u_3\}$ is a basis of U.

---

c) We compute
$$\text{proj}_U (1,-1,2,-1) = \left(\frac{(1,-1,2,-1)\cdot u_1}{\|u_1\|^2}\right) u_1 + \left(\frac{(1,-1,2,-1)\cdot u_2}{\|u_2\|^2}\right) u_2 + \left(\frac{(1,-1,2,-1)\cdot u_3}{\|u_3\|^2}\right) u_3$$

$$= \frac{1}{2}(0,1,1,0) + \frac{-1}{1}(0,0,0,1) + \frac{-3}{2}(0,1,-1,0) = (0,-1,2,-1).$$

---

d) Let $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ & v_4 & & \end{bmatrix}$. We choose $v_4 \in \mathbb{R}^4$ so rank $A = 4$.

$A \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ & v_4 & & \end{bmatrix}$, so if $v_4 = (1,0,0,0)$, rank $A = 4$ so $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$

5(a). State whether the following is true or false, and justify your answer.

- If you say the statement may be false, you <u>must give an explicit example - with numbers, matrices, or functions (as is appropriate), if possible, or an argument using theorems and facts from class.</u>

- If you say the statement is always true, you must give a clear explanation.

(i) Let $\{u, v\}$ be linearly independent in $\mathbf{R}^{2016}$. If A is an invertble 2016×2016 matrix, then $\{Au, Av\}$ is also linearly independent in $\mathbf{R}^{2016}$.

Suppose $aAu + bAv = 0$ for scalars $a, b \in \mathbb{R}$. Then multiplying by A^{-1} (on the left) of both sides yields

$au + bv = 0$. But $\{u, v\}$ is l.i., so $a = b = 0$.

Hence $\{Au, Av\}$ is l.i.

ANSWER $\boxed{\text{TRUE}}$

(ii) The rows of a 3 by 5 matrix are always linearly dependent.

Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Then rank $A = 3$

and the rows of A are thus l.i. (dim row$(A) = 3$)

ANSWER $\boxed{\text{FALSE}}$

5(b). Let A be a $n \times n$ matrix with real entries. Give three additional statements **equivalent to**

"There is $b \in \mathbf{R}^n$ such that the system $Ax = b$ is inconsistent,"

one each in terms of: (So A is <u>not</u> invertible)

(i) The general solution of $Ax = 0$: has at least one parameter;

therefore $Ax=0$ has infinitely many solns.

(ie. the general solnᵘ $= \{x \mid Ax=0\}$ is infinite)

(ii) The column space of A: has dimension $< n$

$(\neq \mathbb{R}^n)$

(iii) the invertibility (or not) of A: A is <u>not</u> invertible

6. [Bonus/Challenge]

Suppose A is an invertible 15×15 matrix and B is any 15×12 matrix with $\operatorname{rank} B = 12$. Prove carefully that $\operatorname{rank} AB = 12$.

(You <u>cannot</u> choose A or B: your proof must work for all A and B satisfying the conditions above.)

Let $B = [b_1 \ldots b_{12}]$ be B in block column form.

We know (since rank $B = 12$) that $\{b_1, \ldots, b_{12}\}$ is l.i.

Now $AB = [Ab_1 \; Ab_2 \; \ldots \; Ab_{12}]$ (ie Ab_j = j^{th} col. of AB)

~~It~~ Suffices to show that $\{Ab_1, \ldots, Ab_{12}\}$ is l.i.,

since rank $AB \le 12$.

Suppose $a_1 Ab_1 + \cdots + a_{12} Ab_{12} = 0$ for scalars $a_1, \ldots, a_{12}$.

Multiplying by A^{-1} on the left of both sides of this equation, we obtain $a_1 b_1 + \cdots + a_{12} b_{12} = 0$.

But $\{b_1, \ldots, b_{12}\}$ is l.i., so $a_1 = a_2 = \cdots = a_{12} = 0$.

Hence $\{Ab_1, \ldots, Ab_{12}\}$ is l.i. and thus

rank $AB = 12$.