F16 MAT 1341A T3 4:00pm V2.

1. For a homogeneous linear system of 301 equations in 289 unknowns, answer the following three questions (in order):

- Can the system have infinitely many solutions? ✓
- Can the system have only one solution? ✓
- Can the system be inconsistent? X

$$\begin{array}{cc} & 289 \\ 301 & \left[\begin{array}{c|c} A & 0 \end{array}\right] \end{array}$$

A. No, No, No.
B. No, No, Yes.
C. Yes, Yes, No.
D. Yes, No, Yes.
E. Yes, No, No.
F. Yes, Yes, Yes.

- We know the system is consistent (0 is a sol^n)
- We know $0 \leq \text{rank} A \leq 289$
  \# variables $= 289$
- " "

So

2. Find the value of t for which $(5, t, 1)$ belongs to span$\{(1, -1, 0), (1, 2, 1)\}$.

A. -2
B. -1
C. 0
D. 1
E. 2
F. 7

Method 1 If $(t, 5, 1) = a(1,-1,0) + b(1,2,1)$ for $a, b \in \mathbb{R}$

then
$$\left.\begin{aligned} t &= a + b \\ 5 &= -a + 2b \\ 1 &= b \end{aligned}\right\} \quad a = -3,\ b = 1\ \&\ t = -2$$

3b). Using part (a), find all values of p and q so that this system has

(i) a unique solution, $\iff \text{rank } A = \text{rank } [A|b] = 3 = \#\text{ vars.}$

$\iff p \neq -1$

(ii) infinitely many solutions, $\iff \text{rank } A = \text{rank } [A|b] < 3 = \#\text{ vars.}$

$\iff p = -1$

$\& \; q = 3$

(iii) no solutions. $\iff \text{rank } A < \text{rank } [A|b]$

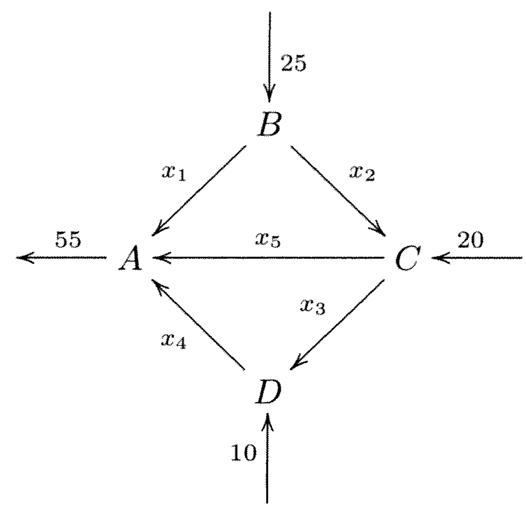
$\iff p = -1 \;\&\; q \neq 3$

3c). In case b(ii) above, give a complete geometric description of the set of solutions.

Then, $[A|b] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$; so $\begin{array}{l} x = -1 + s \\ y = 0 \\ z = s \end{array}$; $s \in \mathbb{R}$

So the gen'l sol'n is $\{(-1,0,0) + s(1,0,1) \mid s \in \mathbb{R}\}$ which is a line through $(-1,0,0)$ in $\mathbb{R}^3$ with dir^n vector $(1,0,1)$.

4. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the traffic flow, together with all the constraints on the variables x_i, $i = 1, \dots, 5$.
(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

Intersection	Flow in	=	Flow out
A	$x_1 + x_4 + x_5$	=	55
B	25	=	$x_1 + x_2$
C	$20 + x_2$	=	$x_3 + x_5$
D	$x_3 + 30$	=	x_4

Constraint $x_i \in \mathbb{Z}$, $i = 1, \dots, 5$ ("# of cars")
& $x_i \geq 0$, $i = 1, \dots, 5$ ("one-way streets")

(Q.4 parts (b) and (c) are on the next page...)

b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\begin{array}{ccccccc} & & & s & t & & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 55 \\ 0 & 1 & 0 & -1 & -1 & -30 \\ 0 & 0 & 1 & -1 & 0 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$\begin{aligned} x_1 &= 55 - s - t \\ x_2 &= -30 + s + t \\ x_3 &= -10 + s \\ x_4 &= s \\ x_5 &= t \end{aligned} \qquad ; s, t \in \mathbb{R}$$

c) If $\overline{CA}$ were closed due to roadwork, **using your results from (b) and the constraints**, find

(i) The maximum flow along $\overline{CD}$, and
(ii) The minimum flow along $\overline{CD}$.

(You must justify all your answers.)

$\overline{CA}$ is closed $\Leftrightarrow$ $x_5 = t = 0$, so, implementing the constraints, we find

$$\left.\begin{array}{llll} x_1 \geq 0 \Leftrightarrow & 55 - s \geq 0 & \Leftrightarrow & 55 \geq s \\ x_2 \geq 0 \Leftrightarrow & -30 + s \geq 0 & \Leftrightarrow & s \geq 30 \\ x_3 \geq 0 \Leftrightarrow & -10 + s \geq 0 & \Leftrightarrow & s \geq 10 \\ x_4 \geq 0 \Leftrightarrow & s \geq 0 & & \end{array}\right\} \Rightarrow 55 \geq s \geq 30$$

$(x_5 = 0 \checkmark)$

Since the flow along $\overline{CD}$ is $x_3 = -10 + s$, we find

(i) the max. flow along $\overline{CD}$ is $-10 + 55 = 45$

(ii) the min. " " " " $-10 + 30 = 20$

3. Suppose $p, q \in \mathbf{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcl} -x & + & y & + & z & = & p \\ x & & & - & z & = & 1 \\ x & + & 2y & + & pz & = & 2p+q \end{array}$$

a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find $\operatorname{rank} A$ and $\operatorname{rank}[\,A\,|\,b\,]$ for <u>all</u> values of p and q.

$$[A|b] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & p \\ 1 & 0 & -1 & 1 \\ 1 & 2 & p & 2p+q \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & p \\ 0 & 1 & 0 & p+1 \\ 0 & 2 & p+1 & 2p+q-1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & p \\ 0 & 1 & 0 & p+1 \\ 0 & 0 & p+1 & q-3 \end{array}\right]$$

So $\operatorname{rank} A = \begin{cases} 2 & \text{if } p = -1 \text{ (all } q) \\ 3 & \text{if } p \neq -1 \text{ (all } q) \end{cases}$

& $\operatorname{rank} [A|b] = \begin{cases} 2, & \text{if } p = -1 \ \& \ q = 3 \\ 3, & \text{otherwise} \end{cases}$

(Q.3 parts (b) and (c) are on the next page...)

5. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit example - with numbers, matrices, or functions (as is appropriate), if possible, or an argument using theorems and facts from class.</u>

- If you say the statement is always true, you must give a clear explanation.

a) Suppose X is a subspace of $\mathbf{R}^{2016}$, that $X \neq \{0\}$ and that X has a spanning set with 1000 vectors. Then, $1 \leq \dim X \leq 1000$.

Since $X \neq 0$, $\dim X \geq 1$. Since $\dim X \leq$ size of any spanning set, $\dim X \leq 1000$.
Hence $1 \leq \dim X \leq 1000$.

ANSWER $\boxed{\text{TRUE}}$

b) If both $m, p > 1$ and an $m \times p$ matrix A has a column of zeros, then $\operatorname{rank} A < p$.

Since every leading one is in a different column and A has a column of zeros then $\operatorname{rank} A \leq p-1 < p$

ANSWER $\boxed{\text{TRUE}}$

6 (cont.).

c) If the coefficient matrix of a linear system of 3 equations in 2 variables has a row of zeros, the system has infinitely many solutions.

e.g. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ has a unique soln

or $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ has no solutions

ANSWER $\boxed{\text{FALSE}}$

d) The coordinate vector of $(3, 2) \in \mathbf{R}^2$ with respect to the ordered basis $\{(1, 2), (1, 1)\}$ is $(1, 1)$.

If this were true,

then $(3,2) = 1(1,2) + 1(1,1)$

$= (2,3)$, which is absurd

Hence this statement is false

ANSWER $\boxed{\text{FALSE}}$