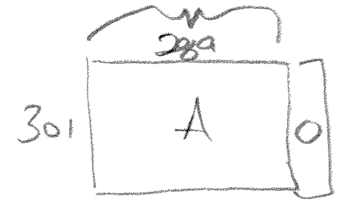
MAT 1341A F16 T3 4:00 V1

1. For a homogeneous linear system of 301 equations in 289 unknowns, answer the following three questions (in order):

- Can the system have infinitely many solutions? ✓
- Can the system have only one solution? ✓
- Can the system be inconsistent? ✗



A. No, No, Yes.
(B) Yes, Yes, No.
C. Yes, No, Yes.
D. Yes, No, No.
E. Yes, Yes, Yes.
F. No, No, No.

- We know the system is consistent (0 is a sol^n)
- $0 \le \text{rank } A \le 289$
- # variables = 289

So

2. Find the value of t for which $(t, 5, 1)$ belongs to span$\{(-1,1,0),(2,1,1)\}$.

(A) -2
B. -1
C. 0
D. 1
E. 2
F. 7

Method 1: If $(t,5,1) = a(-1,1,0) + b(2,1,1)$ for $a, b \in \mathbb{R}$,

then

$$\left.\begin{aligned} t &= -a + 2b \\ 5 &= a + b \\ 1 &= b \end{aligned}\right\} a = 4,\ b = 1,\ \&\ t = -2$$

Method 2: See 2:30 V1 sol^n.

3. Suppose $p, q \in \mathbf{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcl} x & & & - & z & = & 1 \\ -x & + & y & + & z & = & p \\ x & + & 2y & + & pz & = & 2p+q \end{array}$$

a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank A and rank$[\,A\,|\,b\,]$ for <u>all</u> values of p and q.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & p \\ 1 & 2 & p & 2p+q \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & p \\ 0 & 1 & 0 & p+1 \\ 0 & 2 & p+1 & 2p+q-1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & p \\ 0 & 1 & 0 & p+1 \\ 0 & 0 & p+1 & q-3 \end{array}\right].$$

So
$$\text{rank } A = \begin{cases} 2 & \text{if } p = -1 \text{ (all } q) \\ 3 & \text{if } p \neq -1 \text{ (all } q) \end{cases}$$

&
$$\text{rank } [A|b] = \begin{cases} 2, & \text{if } p=-1 \text{ \& } q=3 \\ 3, & \text{otherwise} \end{cases}$$

(Q.3 parts (b) and (c) are on the next page...)

3b). Using part (a), find all values of p and q so that this system has

(i) a unique solution, $\iff$ rank A = rank $[A|b]$ = 3 = # variables

$\iff$

$p \neq -1$

(ii) infinitely many solutions, ~~or~~ $\iff$ rank A = rank $[A|b] < 3$ = # variables

$\iff$ $p = -1$

and $q = 3$

(iii) no solutions. $\iff$ rank A < rank $[A|b]$

$\iff$ $p = -1$ and $q \neq 3$.

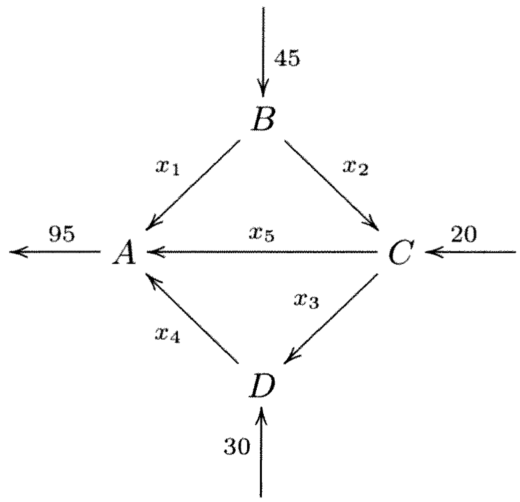
3c). In case b(ii) above, give a complete geometric description of the set of solutions.

Then, $[A|b] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$; so $\begin{aligned} x &= -1 + s \\ y &= 0 \\ z &= s \end{aligned}$; $s \in \mathbb{R}$

So the gen'l sol^n is $\{(-1,0,0) + s(1,0,1) \mid s \in \mathbb{R}\}$

which is a line in $\mathbb{R}^3$ through $(-1,0,0)$ with direction $(1,0,1)$.

4. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the traffic flow, <u>together with all the constraints on the variables</u> x_i, $i = 1, \ldots, 5$.
(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

Intersection	Flow in	=	Flow out
A	$x_1 + x_4 + x_5$	=	95
B	45	=	$x_1 + x_2$
C	$20 + x_2$	=	$x_3 + x_5$
D	$x_3 + 30$	=	x_4

<u>Constraints</u> $x_i \in \mathbb{Z}$, $i = 1, \ldots, 5$ (# of cars)
& $x_i \geq 0$, " (one-way streets)

(Q.4 parts (b) and (c) are on the next page...)

b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{ccccc|c} & & & s & t & \\ 1 & 0 & 0 & 1 & 1 & 95 \\ 0 & 1 & 0 & -1 & -1 & -50 \\ 0 & 0 & 1 & -1 & 0 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$\begin{aligned} x_1 &= 95 - s - t \\ x_2 &= -50 + s + t \\ x_3 &= -30 + s \quad ; \quad s, t \in \mathbb{R} \\ x_4 &= s \\ x_5 &= t \end{aligned}$$

c) If $\overline{CA}$ were closed due to roadwork, **using your results from (b) and the constraints**, find

(i) The maximum flow along $\overline{CD}$, and
(ii) The minimum flow along $\overline{CD}$.

(You must justify all your answers.)

$\overline{CA}$ is closed $\Leftrightarrow x_5 = t = 0$; so, implementing the constraints, we find

$$\left.\begin{aligned} x_1 \geq 0 &\Leftrightarrow 95 - s \geq 0 \Leftrightarrow 95 \geq s \\ x_2 \geq 0 &\Leftrightarrow -50 + s \geq 0 \Leftrightarrow s \geq 50 \\ x_3 \geq 0 &\Leftrightarrow -30 + s \geq 0 \Leftrightarrow s \geq 30 \\ x_4 \geq 0 &\Leftrightarrow s \geq 0 \\ (x_5 = 0 \checkmark) & \end{aligned}\right\} \Rightarrow 95 \geq s \geq 50$$

Since the flow along $\overline{CD}$ is $x_3 = -30 + s$, we find

(i) the max. flow along $\overline{CD}$ is $-30 + 95 = 65$

(ii) the min " " " $-30 + 50 = 20$.

5. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example - with numbers, matrices, or functions (as is appropriate), if possible, or an argument using theorems and facts from class.

- If you say the statement is always true, you must give a clear explanation.

a) Suppose X is a subspace of $\mathbf{R}^{2016}$, that $X \neq \{0\}$ and that X has a spanning set with 1000 vectors. Then, $1 \leq \dim X \leq 1000$.

Since $1 \leq \dim X \leq$ size of any spanning set (Since $X \neq 0$) and we know there's a spanning set with 1000 vectors we can conclude

$1 \leq \dim X \leq 1000$

ANSWER TRUE

b) If both $m, p > 1$ and an $m \times p$ matrix A has a column of zeros, then $\operatorname{rank} A < p$.

(If $\tilde{A}$ is the RRE form of A, $\tilde{A}$ has a column of zeros, since elem. row ops don't mix columns.) Thus, in $\tilde{A}$, ~~there~~ are at most $p-1 < p$ leading ones, so rank $A < p$.

ANSWER TRUE

6 (cont.).

c) If the coefficient matrix of a linear system of 3 equations in 2 variables has a row of zeros, the system has infinitely many solutions.

For example, $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ is the augmented matrix of such a system, which is inconsistent

ANSWER FALSE

d) The coordinate vector of $(3,5) \in \mathbf{R}^2$ with respect to the ordered basis $\{(1,1),(1,2)\}$ is $(1,1)$.

If this were so,

$$(3,5) = 1.(1,1) + 1(1\,2)$$
$$= (3,3),$$ which is absurd.

So this is false

ANSWER FALSE