MAT 1341A F16 T3 2:30 V2

1. Consider a homogeneous system of 2010 linear equations in 1000 unknowns. Which one of the following is true?

1000

2010 [A | 0]

A. The system can be inconsistent.

B. The system can never have infinitely many solutions.

C. The system always has a unique solution.

(D.) The system has either the trivial solution only, or infinitely many solutions.

E. The system has between 1 and 1000 solutions.

F. The system always has infinitely many solutions.

The system is consistent because it is homogeneous.

As well, $0 \leq \text{rank} A \leq 1000$

If rank A = 1000, the system has a unique soln (0)

" < 1000, " " ∞ly many solutions

2. Find the value of t for which $(1, t, 2)$ belongs to $\text{span}\{(0, -1, 1), (1, 2, 1)\}$.

A. -2

B. -1

C. 0

(D.) 1

E. 2

F. 7

Method 1. If $(1, t, 2) = a(0,-1,1) + b(1,2,1)$ for $a, b \in \mathbb{R}$,

then
$$\left.\begin{aligned} 1 &= b \\ t &= -a + 2b \\ 2 &= a + b \end{aligned}\right\} \Rightarrow a = b = 1 \ \& \ t = 1$$

See T3 2:30 V2 for an alternative method (though the question & answer are different there.)

3. Suppose $p, q \in \mathbf{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcl} -x & + & y & + & 2z & = & p \\ x & & & - & z & = & 1 \\ x & + & 2y & + & pz & = & 2p - q \end{array}$$

a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find $\operatorname{rank} A$ and $\operatorname{rank}[\,A\,|\,b\,]$ for all values of p and q.

$$[A|b] = \left[\begin{array}{ccc|c} -1 & 1 & 2 & p \\ 1 & 0 & -1 & 1 \\ 1 & 2 & p & 2p-q \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & p+1 \\ 0 & 2 & p+1 & 2p-q-1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & p+1 \\ 0 & 0 & p-1 & -q-3 \end{array}\right]$$

Hence $\operatorname{rank} A = \begin{cases} 2, & \text{if } p = 1 \text{ (all } q) \\ 3 & \text{if } p \neq 1, \text{ all } q \end{cases}$

& $\operatorname{rank}[A|b] = \begin{cases} 2 & \text{if } p = 1 \ \& \ q = 3 \\ 3, & \text{otherwise} \end{cases}$

(Q.3 parts (b) and (c) are on the next page...)

3b). Using part (a), find all values of p and q so that this system has

(i) a unique solution, $\Leftrightarrow$ $\text{rank } A = \text{rank}[A|b] = 3 = \#\text{variables} \Leftrightarrow$

$p \neq 1$ (all q)

(ii) infinitely many solutions, or $\Leftrightarrow$ $\text{rank } A = \text{rank}[A|b] < 3 = \#\text{variables}$

$\Leftrightarrow$ $p = 1$ & $q = 3$

(iii) no solutions. $\Leftrightarrow$ $\text{rank } A < \text{rank}[A|b]$

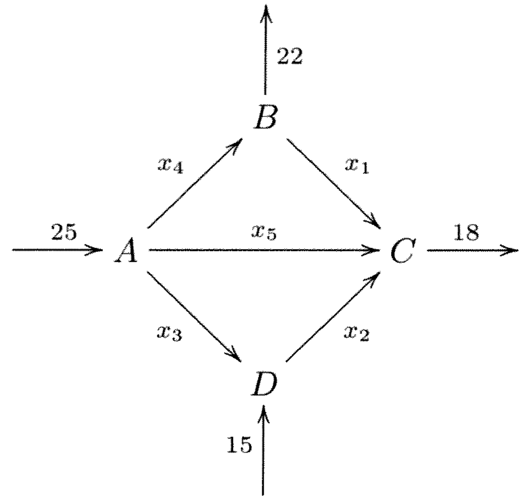
$\Leftrightarrow$ $p = 1$ & $q \neq 3$

3c). In case b(ii) above, give a complete geometric description of the set of solutions.

Then, $[A|b] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$; so $x = 1 + s$, $y = 2 - s$, $z = s$; $s \in \mathbb{R}$

So the general sol^n: $\{(1,2,0) + s(1,-1,1) \mid s \in \mathbb{R}\}$, which is a line in $\mathbb{R}^3$ through $(1,2,0)$ with direction $(1,-1,1)$.

4. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the traffic flow, together with all the constraints on the variables x_i, $i = 1, \ldots, 5$.
(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

Intersection	Flow in	=	Flow out
A	25	=	$x_3 + x_4 + x_5$
B	x_4	=	$22 + x_1$
C	$x_1 + x_2 + x_5$	=	18
D	$15 + x_3$	=	x_2

Constraints: $x_i \in \mathbb{Z}$, $i = 1, \ldots, 5$ (#'s of cars)

$x_i \geq 0$, " . (One way streets)

(Q.4 parts (b) and (c) are on the next page...)

b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & -22 \\ 0 & 1 & 0 & 1 & 1 & 40 \\ 0 & 0 & 1 & 1 & 1 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$x_1 = -22 + s$
$x_2 = 40 - s - t$
$x_3 = 25 - s - t \quad ; s, t \in \mathbb{R}$
$x_4 = s$
$x_5 = t$

c) If $\overline{AC}$ were closed due to roadwork, **using your results from (b) and the constraints**, find

(i) The maximum flow along $\overline{DC}$, and
(ii) The minimum flow along $\overline{DC}$.

(You must justify all your answers.)

$\overline{AC}$ is closed $\Leftrightarrow$ $x_5 = t = 0$, so, implementing the constraints, we find

$x_1 \geq 0 \Leftrightarrow -22 + s \geq 0 \Leftrightarrow s \geq 22$
& $x_2 \geq 0 \Leftrightarrow 40 - s \geq 0 \Leftrightarrow 40 \geq s$
& $x_3 \geq 0 \Leftrightarrow 25 - s \geq 0 \Leftrightarrow 25 \geq s$
& $x_4 \geq 0 \Leftrightarrow s \geq 0$
($x_5 = 0$)

$\Rightarrow 25 \geq s \geq 22$

The flow along $\overline{DC}$ is $x_2 = 40 - s$, so

(i) the max. flow along $\overline{DC}$ is $40 - 22 = 18$
& (ii) " min. " " " is $40 - 25 = 15$

5. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example - with numbers, matrices, or functions (as is appropriate), if possible, or an argument using theorems and facts from class.

- If you say the statement is always true, you must give a clear explanation.

a) Suppose X is a subspace of $\mathbf{R}^{2010}$, that $X \neq 0$ and that $X \neq \mathbf{R}^{2010}$. Then, both 0 and 2010 are possible as the dimension of X.

If $X \neq 0$, then $1 \leq \dim X$

If $X \neq R^{2010}$, then $\dim X < 2009$.

ANSWER FALSE

b) If both $m, p > 1$ and an $m \times p$ matrix A has a row of zeros, then rank $A < p$.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has rank $2 = p$.

$\overset{m}{3} \times \overset{p}{2}$ matrix

ANSWER FALSE

6 (cont.).

c) If the coefficient matrix of a linear system of 3 equations in 4 variables has a column of zeros, the system has infinitely many solutions.

e.g. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ is the augmented matrix of an inconsistent system.

ANSWER FALSE

d) The coordinate vector of $(1, 1) \in \mathbf{R}^2$ with respect to the ordered basis $\{(1, 1), (3, 2)\}$ is $(1, 1)$.

If this were true,

$(1,1) = 1(1,1) + 1(3,2)$

$= (4,3)$, which is absurd.

Hence this is false

ANSWER FALSE