

1. Consider a homogeneous system of 2016 linear equations in 2000 unknowns. Which one of the following is true?

- A. The system can be inconsistent. \times
 B. The system can never have infinitely many solutions. \times
 C. The system has between 1 and 2000 solutions. \times
 D. The system always has infinitely many solutions.
 E. The system always has a unique solution. \times

$$\begin{array}{c} \text{2016} \\ \boxed{A} \end{array} \begin{array}{c} \text{2000} \\ \vdots \\ 0 \end{array} = [A|0]$$

- F. The system has either the trivial solution only, or infinitely many solutions. \checkmark

We know the system is consistent because it is homogeneous. As well, $0 \leq \text{rank } A \leq 2000$.

If $\text{rank } A = 2000$ it has a unique soln (0)
 < 2000 it has infinitely many solns

2. Find the value of t for which $(1, 3, t)$ belongs to $\text{span}\{(0, 1, -1), (1, 1, 2)\}$.

- A. -2
 B. -1
 C. 0
 D. 1
 E. 2
 F. 7

Method 1: If $(1, 3, t) = a(0, 1, -1) + b(1, 1, 2)$
 for $a, b \in \mathbb{R}$,

$$\left. \begin{array}{l} \text{then } 1 = b \\ 3 = a + b \\ \& t = -a + 2b \end{array} \right\} \Rightarrow a = 2, b = 1 \text{ \& so } \underline{t = 0.}$$

Method 2 $(1, 3, t) \in \text{span}\{(0, 1, -1), (1, 1, 2)\}$ iff

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 3 \\ -1 & 2 & t \end{array} \right] \text{ is consistent} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3+t \end{array} \right] \text{ is consistent}$$

$$\Leftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & t \end{array} \right] \text{ is consistent} \Leftrightarrow \underline{t = 0.}$$

(Better for more complicated questions of this kind)

3. Suppose $p, q \in \mathbf{R}$ and consider the linear system in x, y and z :

$$\begin{array}{rclcl} x & & -z & = & 2 \\ -x & + & y & + & z & = & p \\ x & + & 2y & + & pz & = & 2p+q \end{array}$$

a) If $[A|b]$ is the augmented matrix of the system above, find $\text{rank } A$ and $\text{rank}[A|b]$ for all values of p and q .

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ -1 & 1 & 1 & p \\ 1 & 2 & p & 2p+q \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & p+2 \\ 0 & 2 & p+1 & 2p+q-2 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & p+2 \\ 0 & 0 & p+1 & q-6 \end{array} \right]$$

Hence $\text{rank } A = \begin{cases} 2, & \text{if } p = -1, \text{ all } q \\ 3 & \text{if } p \neq -1, \text{ all } q \end{cases}$

$\text{rank}[A|b] = \begin{cases} 2, & \text{if } p = -1 \text{ and } q = 6 \\ 3, & \text{otherwise.} \end{cases}$

3b). Using part (a), find all values of p and q so that this system has

(i) a unique solution, $\Leftrightarrow \text{rank } A = \text{rank } [A \ b] = 3 \Leftrightarrow$

$$p \neq -1, \text{ all } q.$$

(ii) infinitely many solutions, or $\Leftrightarrow \text{rank } A = \text{rank } [A \ b] < 3 = \# \text{ variables}$

$$\Leftrightarrow \begin{cases} p = -1 \\ q = 6 \end{cases}$$

(iii) no solutions $\Leftrightarrow \text{rank } A < \text{rank } [A \ b] \Leftrightarrow$

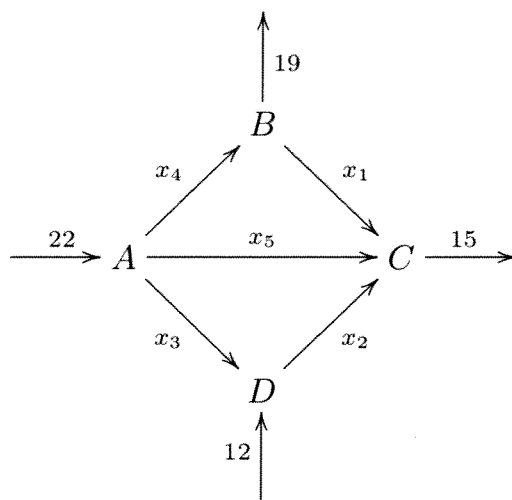
$$\begin{cases} p = -1 \\ q \neq 6 \end{cases}$$

3c). In case b(ii) above, give a complete geometric description of the set of solutions.

In that case, $[A \ b] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right];$ so $\begin{cases} x = 2 + \Delta \\ y = 1 \\ z = \Delta \end{cases}; \Delta \in \mathbb{R}$

So the general soln is $\sum (2, 1, 0) + \Delta(1, 0, 1) \mid \Delta \in \mathbb{R}$, which is a line in \mathbb{R}^3 through $(2, 1, 0)$ with direction vector $(1, 0, 1)$.

4. Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the traffic flow, together with all the constraints on the variables $x_i, i = 1, \dots, 5$.

(Do not perform any operations on your equations: this is done for you in (b). Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.)

Intersection: Flow in = Flow out

$$A \quad 22 = x_3 + x_4 + x_5$$

$$B \quad x_4 = 19 + x_1$$

$$C \quad x_1 + x_2 + x_5 = 15$$

$$D \quad 12 + x_3 = x_2$$

Constraints: $x_i \in \mathbb{Z} \quad i = 1, \dots, 5$
 $x_i \geq 0$
 (one way streets)

(Q.4 parts (b) and (c) are on the next page...)

b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{ccccc|c} \textcircled{1} & 0 & 0 & -1 & 0 & -19 \\ 0 & \textcircled{1} & 0 & 1 & 1 & 34 \\ 0 & 0 & \textcircled{1} & 1 & 1 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$\begin{aligned} x_1 &= -19 + \Delta - t \\ x_2 &= 34 - \Delta - t \\ x_3 &= 22 - \Delta - t \\ x_4 &= \Delta \\ x_5 &= t \end{aligned} \quad ; \Delta, t \in \mathbb{R}$$

c) If \overline{AC} were closed due to roadwork, using your results from (b) and the constraints, find

- The maximum flow along \overline{DC} , and
- The minimum flow along \overline{DC} .

(You must justify all your answers.)

\overline{AC} closed $\Leftrightarrow x_5 = t = 0$; so, implementing the constraints, we find

$$\left. \begin{aligned} x_1 \geq 0 &\Leftrightarrow -19 + \Delta \geq 0 \Leftrightarrow \Delta \geq 19 \\ x_2 \geq 0 &\Leftrightarrow 34 - \Delta \geq 0 \Leftrightarrow 34 \geq \Delta \\ x_3 \geq 0 &\Leftrightarrow 22 - \Delta \geq 0 \Leftrightarrow 22 \geq \Delta \\ x_4 \geq 0 &\Leftrightarrow \Delta \geq 0 \end{aligned} \right\} \Rightarrow 22 \geq \Delta \geq 19$$

($x_5 = 0$)

The flow along \overline{DC} is $x_2 = 34 - \Delta$. Hence

- The max. flow along \overline{DC} is $34 - 19 = 15$
- " " min. " " " " is $34 - 22 = 12$

5. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example - with numbers, matrices, or functions (as is appropriate), if possible, or an argument using theorems and facts from class.
- If you say the statement is always true, you must give a clear explanation.

a) Suppose X is a subspace of \mathbf{R}^{2016} , that $X \neq 0$ and that $X \neq \mathbf{R}^{2016}$. Then, both 0 and 2016 are possible as the dimension of X .

Since $X \neq 0$, $\dim X \geq 1$

Since $X \neq \mathbf{R}^{2016}$, $\dim < 2016$.

ANSWER

FALSE

b) If both $m, p > 1$ and an $m \times p$ matrix A has a row of zeros, then $\text{rank } A < p$.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has rank $2 = p$.
($\begin{matrix} m & p \\ 3 & 2 \\ \text{matrix} \end{matrix}$)

ANSWER

FALSE

6 (cont.).

- c) If the coefficient matrix of a linear system of 2 equations in three variables has a column of zeros, the system has infinitely many solutions.

e.g. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right]$ is the augment matrix of an inconsistent system

ANSWER

FALSE

- d) The coordinate vector of $(1, 1) \in \mathbf{R}^2$ with respect to the ordered basis $\{(1, 1), (2, 1)\}$ is $(1, 1)$.

If this were true, $(1, 1) = 1 \cdot (1, 1) + 1 \cdot (2, 1)$
 $= (3, 2)$, which
 is absurd. Hence this is false.

ANSWER

FALSE

7. [Bonus] If A and B are 3 by 3 matrices with $B \neq 0$ and $AB = 0$, then $\text{rank } A < 3$.

(Your proof must work for all 3 by 3 matrices A and B satisfying the above two conditions: do not choose particular matrices.)

Write $B = [b_1 \ b_2 \ b_3]$ in block column form.

$$\text{Then } AB = [Ab_1 \ Ab_2 \ Ab_3] = [0 \ 0 \ 0]$$

← (in block column form.)

Since $B \neq 0$, at least one column of B is not zero.

If $b_1 \neq 0$, then $Ab_1 = 0$. If

$\text{rank } A = 3$, $[A \mid 0]$ has only the trivial soln, so $Ab_1 = 0$ with $b_1 \neq 0$ is impossible.

Similar arguments apply if b_2 or b_3 (or both) are non-zero. Hence $\text{rank } A < 3$.