1341A.F16 Test 2

V2

1. Which of the following are subspaces of $\mathbf{R}^3$?

$U = \{(x - 2y,\ x + y,\ 3x - y) \mid x,\ y \in \mathbf{R}\}$ = span$\{(1,1,3),(-2,1,-1)\}$, so U is a s.s..
$V = \{(x,\ y^2,\ x + y) \mid x,\ y \in \mathbf{R}\}$ See below
$W = \{(x, y, z) \mid -3x + y = 0\}$ is a plane through the origin and so is a s.s.
$X = \{(x,\ z,\ -z) \mid x,\ z \in \mathbf{R}\}$ = span$\{(1,0,0),(0,1,-1)\}$, so X is a s.s..

A. U and V only
B. U and W only
C. W and X only
D. U, W and X only
E. U, V and X only
F. V and W only

V is not a s.s., since $(0,1,1) \in V$ but $2.(0,1,1) = (0,2,2) \notin V$. So V is not closed under mult. by scalars.

2. Which of the following statements is/are true?

I. The set $\{(1, 2, 3)\}$ spans a line through the origin in $\mathbf{R}^3$. True
II. Any two different vectors in $\mathbf{R}^2$ are linearly independent. False; $\{(1,0),(0,0)\}$ is not l.i.
III. The set $\{(1, 0, 1), (2, 0, 2)\}$ spans $\mathbf{R}^3$. False: dim $\mathbb{R}^3 = 3$.
IV. The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans $\mathbf{M}_{22}$. False: dim $M_{22} = 4$
V. In a vector space V which contains u, v and w, if every vector in V is a linear combination of v and $u + v + w$, then $V = \text{span}\{u, v, w\}$. True:

If $v = au + b(u+v+w)$ then $v = (a+b)u + bv + bw$, so $\{u,v,w\}$ spans V.

A. I only
B. II only
C. II and III only
D. I and V only
E. I, II, IV, and V only
F. All of the statements are true.

3. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x + 2y = 0\}$.

a) Explain *very briefly* why W is a subspace of $\mathbf{R}^3$. (*You will not need to use the Subspace Test.*)

b) Find a spanning set for W.

c) Find a basis for W.

d) Give a complete geometric description of W.

(Remember that you must justify your answers.)

a) W is a plane through O in $\mathbb{R}^3$ $\therefore$ W is a s.s. of $\mathbb{R}^3$

b) Note that $(x,y,z) \in W \iff x = -2y$ so

$$W = \{(-2y, y, z) \mid y, z \in \mathbb{R}\} = \text{span}\{\underbrace{(-2,1,0)}_{v_1}, \underbrace{(0,0,1)}_{v_2}\}$$

So $\{v_1, v_2\}$ spans W.

c) Since neither v_1 nor v_2 is a multiple of the other, $\{v_1, v_2\}$ is l.i. (OR: $yv_1 + zv_2 = (0,0,0) \Rightarrow (-2y, y, z) = (0,0,0) \Rightarrow y = z = 0$) Hence $\{v_1, v_2\}$ spans W and is l.i. so $\{v_1, v_2\}$ is a basis of W.

d) W is the plane through O with normal $(1,2,0)$.

4. Let $\mathbf{M}_{2\,2}$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \underbrace{\begin{bmatrix} a & b \\ b & a+c \end{bmatrix}}_{a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \in \mathbf{M}_{2\,2} \mid a, b, c \in \mathbf{R} \right\}.$$

a) Either check that U is closed under addition, or express U in another form so you can simply state a theorem that guarantees that U is a subspace.

(For parts (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{2\,2}$.)

b) Find a basis for U, and hence find $\dim U$.

c) Give a basis for U, different from the one you gave in (b).

(Remember that you must justify your answers.)

a) Note that $U = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_3} \right\}$, and so U is a subspace of M_{22}.

b) We know $\{A_1, A_2, A_3\}$ spans U by (a). Suppose scalars a, b, c satisfy $aA_1 + bA_2 + cA_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$aA_1 + bA_2 + cA_3 = \begin{bmatrix} a & b \\ b & a+c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which implies $a = 0$, $b = 0$ and $a + c = 0$. The latter implies $a = b = c = 0$. Hence $\{A_1, A_2, A_3\}$ is l.i., and thus is a basis of U. Thus $\dim U = 3$.

c) Let $B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$\begin{bmatrix} a & b \\ b & a+c \end{bmatrix} = \frac{a}{2}B_1 + \frac{b}{2}B_2 + \frac{c}{2}B_3$, so $U = \text{span}\{B_1, B_2, B_3\}$.

If $xB_1 + yB_2 + zB_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} 2x & 2y \\ 2y & 2x+2z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which implies $x = y = z = 0$. Hence $\{B_1, B_2, B_3\}$ is also l.i., and thus is a basis for U.

5. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example - with numbers, or functions, as is appropriate!

- If you say the statement is always true, you must give a clear explanation.

a) $X = \{f \in \mathbf{F}(\mathbf{R}) \mid f(x) \geq -1 \quad \text{for all } x \in \mathbf{R}\}$ is a subspace of $\mathbf{F}(\mathbf{R})$. This is false. Let $f(x) = -1$ for all $x \in \mathbb{R}$ be the constant function whose value is -1 for all $x \in \mathbb{R}$. Then $f \in X$, but $2f \notin X$, since $2f(x) = -2 \ngeq -1$ (for all $x \in \mathbb{R}$). Hence X is not a subspace.

ANSWER $\boxed{\text{FALSE}}$

b) If V is a vector space and $\{v_1, v_2, v_3\} \subset V$ spans V, then $\{v_1, v_2, v_3\} \subset V$ is linearly independent.

This is false; here's an example. Let $V = \mathbb{R}^2$ and $v_1 = (1,0)$, $v_2 = (0,1)$ & $v_3 = (0,0)$. Then $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^2$ ($(a,b) = av_1 + bv_2 + 0v_3, \forall a,b \in \mathbb{R}$) but $\{v_1, v_2, v_3\}$ is not l.i. because it contains the zero vector.

ANSWER $\boxed{\text{FALSE}}$

5 (cont.).

$Y =$

c) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22} \mid a + c = 0 \right\}$ is a subspace of $\mathbf{M}_{22}$.

Note $Y = \left\{ \begin{bmatrix} a & b \\ -a & d \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$,

and hence Y is a subspace of M_{22}.

ANSWER $\boxed{\text{TRUE}}$

d) If u_1, u_2 and u_3 are vectors in a vector space U, and $\{u_1, u_2, u_3\}$ spans U, then $\dim U = 3$.

This is false : see the example in (b).

If $U = \mathbb{R}^2$, $\{(1,0), (0,1), (0,0)\}$ spans $\mathbb{R}^2$ but we know $\dim \mathbb{R}^2 = 2 < 3$.

ANSWER $\boxed{\text{FALSE}}$

6. [Bonus] Suppose that u, v, w are non-zero vectors in $\mathbf{R}^{2016}$ such that $u \cdot v = u \cdot w = v \cdot w = 0$. Prove that $\{u, v, w\}$ is linearly independent.

(Your proof must work for *all* choices of non-zero vectors in $\mathbf{R}^{2016}$ such that $u \cdot v = u \cdot w = v \cdot w = 0$ — do not choose the vectors yourself. Use the definition. No 'geometric' argument - e.g. "they are not co-planar" - will suffice, and in any case is meaningless to low-dimensional beings like your instructor and marker.)

Please see the other version for a solution.