1341A. F16 Test 2- V1

1. Which of the following are subspaces of $\mathbf{R}^3$?

$U = \{(x-y,\ x+y,\ x-y) \mid x,\ y \in \mathbf{R}\}$ = span$\{(1,1,1), (-1,1,-1)\}$ $\therefore$ U is a s.s.

$V = \{(x,\ y,\ -y) \mid x,\ y \in \mathbf{R}\}$ = span$\{(1,0,0), (0,1,-1)\}$ $\therefore$ V is a s.s.

$W = \{(x^2,\ y,\ x+y) \mid x,\ y \in \mathbf{R}\}$

$X = \{(x,y,z) \mid x-y=0\}$ is a plane through the origin & so is a s.s.

A. U and V only
B. U and W only
C. W and X only
D. U, W and X only
E. U, V and X only
F. V and W only

From the above & the answers below, we know (D) is correct.

To see that W is not a s.s, we note $(1,0,1) \in W$ but $2(1,0,1) = (2,0,2) \notin W$, so W is not closed under multiplication by scalars.

For, $(1,0,1)$ and $(1,0,1) \in W$ but their sum $(2,0,2) \notin W$, so W is not closed under add^n.

2. Which of the following statements is/are true?

I. The span of any two different vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$. F: span$\{(1,0), (0,0)\} \neq \mathbb{R}^2$

II. The set $\{(1,2)\}$ spans a line through the origin in $\mathbf{R}^2$. True

III. In a vector space V which contains u, v and w, if every vector in V is a linear combination of u and $u+v+w$, then $V = \text{span}\{u,v,w\}$. True: $au + b(u+v+w)$ $= (a+b)u + bv + bw$

IV. The set $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ spans $\mathbf{M}_{22}$. False $\because$ dim $M_{22} = 4 > 2$

V. The set $\{(1,0,1),(0,2,3)\}$ spans $\mathbf{R}^3$. False: dim $\mathbb{R}^3 = 3 > 2$

A. I only
B. II only
C. II and III only
D. IV and V only
E. I, II, IV, and V only
F. All of the statements are true.

3. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x + y - z = 0\}$.

a) Explain *very briefly* why W is a subspace of $\mathbf{R}^3$. (*You will not need to use the Subspace Test.*)

b) Find a spanning set for W.

c) Find a basis for W.

d) Give a complete geometric description of W.

(Remember that you must justify your answers.)

a) W is a plane through the origin in $\mathbb{R}^3$ and $\therefore$ is a subspace of $\mathbb{R}^3$.

b) $W = \{(-y+z, y, z) \mid y, z \in \mathbb{R}\}$ (as $x = -y+z$ iff $(x,y,z) \in W$)

$= \text{span}\{(-1,1,0), (1,0,1)\}$

$\therefore \{\underbrace{(-1,1,0)}_{v_1}, \underbrace{(1,0,1)}_{v_2}\}$ spans W.

c) We know from (b) that $\{v_1, v_2\}$ as above span V, Since neither v_1 nor v_2 is a multiple of the other, $\{v_1, v_2\}$ is l.i.. Hence $\{v_1, v_2\}$ is a basis for W.

OR: If $av_1 + bv_2 = (0,0,0)$, then $\left.\begin{array}{l} -a+b=0, \\ a = 0 \\ b = 0 \ \& \end{array}\right\} \Rightarrow a = b = 0.$

Hence $\{v_1, v_2\}$ is l.i.

d) W is the plane through $(0,0,0)$ with normal $(1,1,-1)$.

4. Let $\mathbf{M}_{22}$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \in \mathbf{M}_{22} \mid a, b, c \in \mathbf{R} \right\}.$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

a) Either check that U is closed under addition, or express U in another form so you can simply state a theorem that guarantees that U is a subspace.

(For parts (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{22}$.)

b) Find a basis for U, and hence find $\dim U$.

c) Give a basis for U, different from the one you gave in (b).

(Remember that you must justify your answers.)

a) Note that $U = \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_3} \right\}$ and so is a s.s. of M_{22}.

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b) By (a), $\{A_1, A_2, A_3\}$ spans U. If $aA_1 + bA_2 + cA_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $a = b = c = 0$. Hence $\{A_1, A_2, A_3\}$ is l.i., and so is a basis for U.

Thus $\dim U = 3$.

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c) Note that $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = 2a \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}}_{B_1} + 2b \underbrace{\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}}_{B_2} + 2c \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}}_{B_3}$,

$\{B_1, B_2, B_3\}$ spans U. Moreover, if $c_1 B_1 + c_2 B_2 + c_3 B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

then $\begin{bmatrix} \frac{c_1}{2} & \frac{c_2}{2} \\ -\frac{c_2}{2} & \frac{c_3}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $c_1 = c_2 = c_3 = 0$. Hence $\{B_1, B_2, B_3\}$ is l.i., and so is a basis for U.

5. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you must give an explicit example - with numbers, or functions, as is appropriate!
- If you say the statement is always true, you must give a clear explanation.

a) $X = \{f \in \mathbf{F}(\mathbf{R}) \mid f(x) \leq 0 \quad \text{for all } x \in \mathbf{R}\}$ is a subspace of $\mathbf{F}(\mathbf{R})$ This is false: let $f(x) = -1$ for all $x \in \mathbb{R}$ be the constant function whose value is -1 for all x. Then $f \in X$, but $(-f) \notin W$, since $-f(x) = 1 \nleq 0$ for all $x \in \mathbb{R}$.

ANSWER $\boxed{\text{FALSE}}$

b) If V is a vector space and $\{v_1, v_2\}$ spans V, then $\{v_1, v_2, v_3\}$ spans V for *any* vector $v_3 \in V$.

This is true: if $\{v_1, v_2\}$ spans V, then for any vector $v \in V$, $v = av_1 + bv_2 + 0\,v_3$ for some scalars a, b, c. Hence $\{v_1, v_2, v_3\}$ spans V

ANSWER $\boxed{\text{TRUE}}$

5 (cont.).

c) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22} \mid b + d = 0 \right\}$ is a subspace of $\mathbf{M}_{22}$.

Note: $W = \left\{ \begin{bmatrix} a & b \\ c & -b \end{bmatrix} \mid a, b, c \right\}$

$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$, and

hence W is a subspace of M_{22}.

ANSWER $\boxed{\text{TRUE}}$

d) If u_1, u_2 and u_3 are vectors in a vector space U, and $\{u_1, u_2, u_3\}$ is linearly independent, then $\dim U = 3$.

This is false in the case $\dim U \geq 4$. For example,

let $U = \mathbb{R}^4$, $u_1 = (1,0,0,0)$, $u_2 = (0,1,0,0)$ & $u_3 = (0,0,0,1)$

Then $\{u_1, u_2, u_3\}$ is l.i. ($au_1 + bu_2 + cu_3 = (0,0,0,0) \Rightarrow$

$(a, b, c, 0) = (0,0,0,0) \Rightarrow a = b = c = 0$.) but $(0,0,0,1) \in \mathbb{R}^4$

while $(0,0,0,1) \notin \text{Span}\{u_1, u_2, u_3\} = \{(a,b,c,0) \mid a, b, c \in \mathbb{R}\}$

ANSWER $\boxed{\text{FALSE}}$

6. [Bonus] Suppose that u, v, w are non-zero vectors in $\mathbf{R}^{2016}$ such that $u \cdot v = u \cdot w = v \cdot w = 0$. Prove that $\{u, v, w\}$ is linearly independent.

(Your proof must work for *all* choices of non-zero vectors in $\mathbf{R}^{2016}$ such that $u \cdot v = u \cdot w = v \cdot w = 0$ — do not choose them yourself. Use the definition. No 'geometric' argument - e.g. "they are not co-planar" - will suffice, and in any case is meaningless to low-dimensional beings like your instructor and marker.)

Suppose $a, b, c \in \mathbb{R}$ and $au + bv + cw = 0$. $(*)$

Then, taking dot products of both sides with u, we obtain

$$a(u \cdot u) + b\underbrace{(u \cdot v)}_{=0} + c\underbrace{(u \cdot w)}_{=0} = u \cdot 0 = 0$$

$$\Rightarrow \quad a\|u\|^2 = 0 \ (**)$$ Since $u \neq 0$, $\|u\| \neq 0$, so $(**)$ implies $a = 0$.

Taking dot product with v & w (resp.) with $(*)$ shows in the same way that b & c (resp.) are zero as well.

Hence $au + bv + cw = 0 \Rightarrow a = b = c = 0$, so $\{u, v, w\}$ is l.i.