MAT 1341A F16 Final Solns



**1.** Which of the following are subspaces of $\mathbf{R}^3$?

$U = \{(x-y,\ x+y,\ x-y) \mid x,\ y \in \mathbf{R}\}$ $= \text{span}\{(1,1,1), (-1,1,-1)\}$ & $\therefore$ is a S.S.
$V = \{(x,\ y,\ -y) \mid x,\ y \in \mathbf{R}\}$ $= \text{span}\{(1,0,0), (0,1,-1)\}$ & $\therefore$ is a S.S.
$W = \{(x^2,\ y,\ x+y) \mid x,\ y \in \mathbf{R}\}$ See below *
$X = \{(x, y, z) \mid x - y = 0\}$ is a plane through $(0,0,0)$ (with normal $(1,-1,0)$) & $\therefore$ is a S.S.

A. $U$ and $V$ only
B. $U$ and $W$ only
C. $W$ and $X$ only
D. $U$, $W$ and $X$ only
E. $U$, $V$ and $X$ only
F. $V$ and $W$ only

* As $U, V, X$ are S.S. and $U, V, W, X$ is not a possible answer, E. must be correct.

Note: $(1,0,1) \in W$ but $2.(1,0,1) = (2,0,2) \notin W$, so $W$ is not. a S.S. of $\mathbb{R}^3$

**2.** Let $\mathbf{M}_{22}$ denote, as usual,the vector space of real $2 \times 2$ matrices, and $A^t$ denote the transpose of $A \in \mathbf{M}_{22}$. The dimension of $V = \{A \in \mathbf{M}_{22} \mid A = A^t\}$ is:

A. 0

B. 1

C. 2

D. 3

E. 4

F. $V$ is not a subspace of $\mathbf{M}_{22}$, so we cannot speak of its dimension.

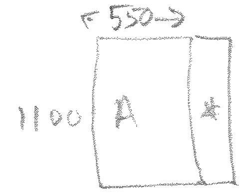
$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\} = \text{span}\left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{M_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

Moreover $aM_1 + bM_2 + cM_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\Rightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so

$\{M_1, M_2, M_3\}$ is a basis for $V$.

Hence $\dim V = 3$

**3.** Complete the following phrase to make a true statement:




"A system of 1100 linear equations in 550 unknowns..."

A. ... always has a solution.

B. ... always has a unique solution.

C. ... may be inconsistent.

D. ... which is consistent always has a unique solution.

E. ... which is consistent never has a unique solution.

F. ... is never consistent.

eg $\left[\begin{array}{c|c} 0 & \begin{matrix}1\\0\\ \vdots \\0\end{matrix}\end{array}\right]$ is inconsistent.

**4.** If $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ then $A^{-1} =$ $\frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$

A. $\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$ B. $\begin{bmatrix} 1/2 & -1/2 \\ -5 & 2 \end{bmatrix}$ C. $\begin{bmatrix} 1/2 & -5/2 \\ 2 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 0 & -1 \\ 1/2 & 3/2 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$ F. $\begin{bmatrix} -1 & -3/2 \\ 5/2 & 2 \end{bmatrix}$

**5.** If $c_j$ denotes the $j^{\text{th}}$ column of

$$A = \begin{bmatrix} 1 & 4 & -1 & -2 \\ 0 & 0 & 0 & 5 \\ 2 & 8 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(j = 1, \ldots 4)$, which of the following sets is a basis for the column space of $A$ ?

$$\sim \begin{bmatrix} 1 & 4 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A. $\{c_1\}$
B. $\{c_1,\, c_2\}$
C. $\{c_1,\, c_3\}$
D. $\{c_1,\, c_4\}$
E. $\{c_1,\, c_2,\, c_3\}$
F. $\{c_1,\, c_3,\, c_4\}$

**6.** Suppose $A$ is an $n \times n$ matrix. Among the following statements, which one is **not equivalent** to the others?

A. $A$ is not invertible.

B. $Ax = 0$ has infinitely many solutions $x \in \mathbf{R}^n$.

C. There is $b \in \mathbf{R}^n$ such that $Ax = b$ is inconsistent.

D. The determinant of $A$ is zero.

E. $A$ is row-equivalent to the identity matrix. ← This would imply A was invertible

F. The rank of $A$ is less than $n$.

**7.** The set of vectors $\{(0,3,4),(1,0,0),(0,4,-3)\}$ is orthogonal. Find the Fourier coefficients $(a_1,a_2,a_3)$ of $v=(0,-1,-1)$, i.e. find real scalars $(a_1,a_2,a_3)$ such that $v=a_1(0,3,4)+a_2(1,0,0)+a_3(0,4,-3)$.

A. $(-\frac{7}{25},0,-\frac{1}{25})$

B. $(-\frac{7}{5},0,-\frac{1}{5})$

C. $(-7,0,-1)$

D. $(0,-1,-1)$

E. $(\frac{7}{25},0,\frac{1}{25})$

F. $(-\frac{7}{25},0,\frac{1}{25})$

$$a_1 = \frac{(0,3,4)\cdot(0,-1,-1)}{(0^2+3^2+4^2)} = \frac{-7}{25}$$

$$a_2 = \frac{(1,0,0)\cdot(0,-1,-1)}{1^2} = 0$$

$$a_3 = \frac{(0,4,-3)\cdot(0,-1,-1)}{25} = \frac{-1}{25}$$

**8.** If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = 7$, find $\begin{vmatrix} 3a-5g & 3b-5h & 3c-5j \\ g & h & j \\ d & e & f \end{vmatrix} \overset{5R_2+R_1 \to R_1}{=} \begin{vmatrix} 3a & 3b & 3c \\ g & h & j \\ d & e & f \end{vmatrix}$

A. 7

B. 21

C. -21

D. 35

E. -35

F. -7

$$= 3\begin{vmatrix} a & b & c \\ g & h & j \\ d & e & f \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{=} -3\cdot\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix}$$

$$= -3\cdot 7 = -21$$

18:50

**9.** Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. Answer the following questions:

- ∘ Is 2 the only eigenvalue of $A$? $|A - \lambda I| = (2-\lambda)^2$, so YES
- ∘ Is the dimension of $\ker(A - 2I)$ equal to 2? $\ker A - 2I = \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has dim $= \underset{(\#\text{cols})}{2} - \underset{(\text{rank})}{1} = 1$ NO
- ∘ Is $A$ diagonalizable? NO, since $\dim E_2 < 2$, and 2 is the only eval.

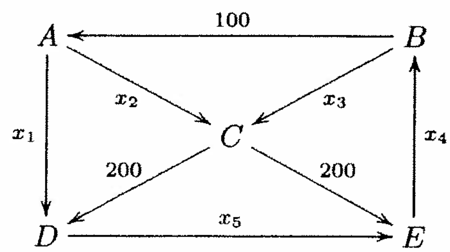
A. Yes, Yes, Yes.
B. Yes, Yes, No.
C. Yes, No, Yes.
D. No, Yes, Yes.
E. No, Yes, No.
F. Yes, No, No.

**10.** Which of the following statements are (always) true?

(1) Each spanning set for $\mathbf{R}^n$ has exactly $n$ vectors. ✗
(2) If $\{u, v, w\}$ is linearly independent, then $\{u, v\}$ is also linearly independent. ✓
(3) If $A$ is an $n \times n$ matrix, then $\det(-A) = -\det(A)$. (✗; only if $n$ is odd)
(4) If $A$ is an $n \times n$ matrix, then $\dim \operatorname{col} A = n$. ✗
(5) If $A$ is an $n \times n$ matrix, the $\dim \ker A = n - \operatorname{rank} A$. ✓ (Thm seen in class)
(6) $\{1, \sin x, \cos x\}$ is linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$. ✓ (Seen in class)

A. All six are true.
B. (2), (5) and (6).
C. (1), (2) and (4).
D. (3), (2) and (6).
E. (4), (5) and (6).
F. (2), (3) and (5).

**11.** Consider the network of streets and intersections below. The arrows indicate the direction of traffic flow along the <u>one-way streets</u>, and the numbers refer to the <u>exact number</u> of cars observed to enter or leave the intersections during one minute. Each $x_i$ denotes the unknown number of cars which passed along the indicated streets during the same period.




a) Write down a system of linear <u>equations</u> which describes the the traffic flow, **together with all the constraints** on the variables $x_i$, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

| Intersection | Flow in | = | Flow out |
|---|---|---|---|
| A | $100$ | = | $x_1 + x_2$ |
| B | $x_4$ | = | $100 + x_3$ |
| C | $x_2 + x_3$ | = | $200 + 200$ |
| D | $x_1 + 200$ | = | $x_5$ |
| E | $x_5 + 200$ | = | $x_4$ |

2

Constraints: $x_i \in \mathbb{Z}$ $(i = 1, \ldots, 5)$ 1/2

$x_i \geq 0$ " . 1/2

**11. b)** The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{ccccc}
 & & & & s \\
\end{array}$$
$$\left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & -1 & -200 \\
0 & 1 & 0 & 0 & 1 & 300 \\
0 & 0 & 1 & 0 & -1 & 100 \\
0 & 0 & 0 & 1 & -1 & 200 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Give the general solution. (Ignore the constraints at this point.)

$x_1 = -200 + s$
$x_2 = 300 - s$
$x_3 = 100 + s$ ; $s \in \mathbb{R}$
$x_4 = 200 + s$
$x_5 = s$

2

**11. c)** Find the maximum and minimum flow along BC, **using your results from (b).**
*(You must justify all your answers.)*

Flow along $\overline{BC}$ is $x_3 = 100 + s$

Implementing constraints

$x_1 \geq 0 \iff -200 + s \geq 0 \iff s \geq 200$
$x_2 \geq 0 \iff 300 - s \geq 0 \iff 300 \geq s$
$x_3 \geq 0 \iff 100 + s \geq 0 \iff s \geq -100$
$x_4 \geq 0 \iff 200 + s \geq 0 \iff s \geq -200$
$x_5 \geq 0 \iff s \geq 0$

$\iff 300 \geq s \geq 200$

Hence the minimum flow along $\overline{BC}$ is $100 + 200 = 300$
max. " " " $100 + 300 = 400$

$\frac{1}{2} + \frac{1}{2}$

**12.** Let

$$A = \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \end{array} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ast$$

a) Find a basis for the column space col $(A)$ of $A$.

b) Find a basis for ker$(A)$.

c) Give a complete geometric description of ker$(A)$.

d) Extend your basis of col $(A)$ to a basis of $\mathbf{R}^4$.

a) By the col. space algorithm and $\ast$, $\{c_1, c_2\}$ is a basis for col$(A)$.

b) $[A \mid 0] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad \begin{array}{l} x = s \\ y = -3s/2 \\ z = s \end{array}$ ; $s \in \mathbb{R}$

$\therefore$ by the kernel algorithm, $\{(1, -\frac{3}{2}, 1)\}$ is a basis for ker $A$.

c) From (b), we see that ker $A$ is the line in $\mathbb{R}^3$ through $(0,0,0)$ with direction $(1, -\frac{3}{2}, 1)$.

d) We seek 2 vectors $w_3$ & $w_4$ s.t.

$$\text{rank} \underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ & w_3 & & \\ & w_4 & & \end{bmatrix}}_{B} = 4. \quad \text{But } B \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ & w_3 & & \\ & w_4 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ & w_3 & & \\ & w_4 & & \end{bmatrix}$$

Hence if $w_3 = (0,0,1,0)$ & $w_4 = (0,0,0,1)$ rank $B = 4$, so $\{c_1, c_2, w_3, w_4\}$ is our basis.

**13.** Let $U = \{(x,y,z,w) \in \mathbf{R}^4 \mid x+z+w=0\}$. $= \ker [1\ 0\ 1\ 1]$

a) Find a basis of $U$ and give the dimension of $U$.

b) Find an orthogonal basis of $U$.

c) Find the best approximation to $(1,1,0,0)$ by vectors in $U$.

a) $U = \ker[1\ 0\ 1\ 1]$ ; $[\,1\ \overset{r}{0}\ \overset{s}{1}\ \overset{t}{1} \mid 0\,]$ $\therefore$ $x = -s-t$, $y = r$, $z = s$, $w = t$ ; $r,s,t \in \mathbb{R}$

$\therefore$ by the kernel algorithm, $\{\overset{v_1}{(0,1,0,0)}, \overset{v_2}{(-1,0,1,0)}, \overset{v_3}{(-1,0,0,1)}\}$ is a basis for $U$. Hence $\dim U = 3$

b) Apply G-S: $u_1 = v_1 = (0,1,0,0)$

$u_2 = v_2 - \dfrac{\cancel{v_2 \cdot u_1}^{\,0}}{\|u_1\|^2} u_1 = u_2 = (-1,0,1,0)$

$u_3 = v_3 - \left(\dfrac{\cancel{v_3 \cdot u_1}^{\,0}}{\|u_1\|^2}\right) u_1 - \left(\dfrac{v_3 \cdot u_2}{\|u_2\|^2}\right) \cdot u_2$

$= (-1,0,0,1) - \dfrac{1}{2}(-1,0,1,0)$

$= (-\tfrac{1}{2}, 0, -\tfrac{1}{2}, 1)$. We rescale this

to $\tilde{u}_3 = (-1,0,-1,2)$ $\therefore$ $\{u_1, u_2, \tilde{u}_3\}$ is an orthogonal basis of $U$.

c) We compute $\operatorname{proj}_U (1,1,0,0) = \dfrac{(1,1,0,0)\cdot(0,1,0,0)}{1} \cdot (0,1,0,0)$

$+ \dfrac{(1,1,0,0)\cdot(-1,0,1,0)}{2}(-1,0,1,0) + \dfrac{(1,1,0,0)\cdot(-1,0,-1,2)}{6} \cdot (-1,0,-1,2)$

$= (0,1,0,0) - \frac{1}{2}(-1,0,1,0) - \frac{1}{6}(-1,0,-1,2) = (\frac{2}{3}, 1, -\frac{1}{3}, -\frac{1}{3})$

$= \frac{1}{3}(2,1,-1,-1)$.

**14.** Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

a) Find the characteristic polynomial $\det(A - \lambda I)$ of $A$, factor it, and deduce that the eigenvalues of $A$ are 1 and 2.

b) Find a basis of $E_1 = \{x \in \mathbf{R}^3 \mid Ax = x\}$.

c) Find a basis of $E_2 = \{x \in \mathbf{R}^3 \mid Ax = 2x\}$.

d) Find an invertible matrix $P$ such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix $D$. Explain why your choice of $P$ is invertible.

a) $|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} \overset{\text{col } 2}{=} (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)\{\lambda^2 - 3\lambda + 2\}$

$= (2-\lambda)(\lambda - 1)(\lambda - 2)$

$= 0 \iff \lambda = 2$ or $\lambda = 1$. Hence evals are 1 & 2.

b) $E_1 = \ker[A - I] = \ker \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \ker \begin{array}{c} \begin{array}{ccc} & & s \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \{(-2s, s, s) \mid s \in \mathbb{R}\}$

$\therefore \{(-2, 1, 1)\}$ is basis for $E_1$

c) $E_2 = \ker[A - 2I] = \ker \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \ker \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \{(-t, s, t) \mid s, t \in \mathbb{R}\}$

$\therefore \{(-1, 0, 1), (0, 1, 0)\}$ is a basis for $E_2$.

d) Set $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then

$P$ is invertible (because $\dim E_1 + \dim E_2 = 3$; or $\det P = \begin{vmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{vmatrix}$

$\overset{\text{row } 3}{=} -1 \cdot \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$), and $P^{-1}AP = D$.

**15.** State whether the following are true or false. You must give reasons for your answer.

(a) If $u, v$ and $w$ are three linearly independent vectors in a vector space $E$, then $u \in \operatorname{span}\{u - v, v - w\}$.

$\boxed{\text{FALSE}}$

Suppose $u = a(u-v) + b(v-w)$ for $a, b \in \mathbb{R}$.

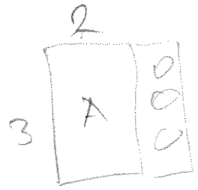
Then $0 = (a-1)u + (b-a)v - bw$. (*)

Since $\{u, v, w\}$ is l.i., (*) implies $a = 1$, $b = a$ & $b = 0$, which is impossible. Hence no such $a, b \in \mathbb{R}$ exist.

Thus $u \notin \operatorname{span}\{u-v, v-w\}$

(b) Let $A$ be a real $3 \times 2$ matrix and suppose there is a vector $v \in \mathbf{R}^2$ with $v \neq 0$ and $Av = 0$. Then the columns of $A$ are linearly dependent.

$\boxed{\text{TRUE}}$




If $A = [c_1\ c_2]$ and $v = (a, b) \neq (0, 0)$

then $ac_1 + bc_2 = 0$ but not both $a, b$ are zero. Hence $\{c_1, c_2\}$ is dependent.

OR: Since $v \neq 0$ satisfies $Av = 0$, rank $A < 2$. Hence $\dim \operatorname{col}(A) < 2$, so the columns of $A$ are linearly dependent.

c) Let $A$ be a $3 \times 4$ matrix. Then every vector in the kernel (or nullspace) of $A$ is orthogonal to every row of $A$.

$\boxed{\text{TRUE}}$

Write $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ block row form. and

let $v \in \ker A$. Then $Av = 0$, But

$$Av = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} v = \begin{bmatrix} r_1 \cdot v \\ r_2 \cdot v \\ r_3 \cdot v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

So yes every vector in $\ker A$ is orthogonal to every row of $A$.

d) If $A$ is an invertible $2 \times 2$ matrix, then $A$ is diagonalizable.

$\boxed{\text{FALSE}}$

(See Q. 9!) If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then

2 is the only eval of $A$ but $\dim E_2 = \dim \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $= 1 < 2$. So $A$ is not diag'ble. However, $\det A = 4 \neq 0$ so $A$ is invertible.

**16. (3 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.**

In parts (a) and (b), $A$ denotes a symmetric $n \times n$ matrix, i.e., $A = A^t$.
(N.B. Your proofs must be valid for *every* symmetric $n \times n$ matrix $A$, so don't choose $n$ or $A$ !)

a) Prove that $(Au) \cdot v = u \cdot (Av)$ for all $u, v \in \mathbf{R}^n$, where " $\cdot$ " denotes the dot product.

Note that $\forall\ x, y \in \mathbb{R}^n$, $\underbrace{x \cdot y}_{\text{dot product}} = \underbrace{x^t y}_{\text{matrix product}}$

Thus, $(Au) \cdot v = \underbrace{(Au)^t v}_{\text{matrix product}} = u^t A^t v = u^t A v = u \cdot (Av)$

(Since $A^t = A$)

Now let $v_\lambda \in \mathbf{R}^n$ be an eigenvector of $A$ with eigenvalue $\lambda$, and set $W = \{w \in \mathbf{R}^n \mid w \cdot v_\lambda = 0\}$.

b) Prove that if $w \in W$, then $Aw \in W$.

If $w \cdot v_\lambda = 0$, then $(Aw) \cdot v_\lambda = w \cdot (Av_\lambda)$ (by a

$= w \cdot (\lambda v_\lambda)$

$= \lambda (w \cdot v_\lambda)$

$= 0$

Hence $Aw \in W$.