NUCLEAR DIMENSION, $\mathcal{Z}$-STABILITY, AND ALGEBRAIC SIMPLICITY FOR STABLY PROJECTIONLESS C^*-ALGEBRAS

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Abstract. The main result here is that a separable C^*-algebra is $\mathcal{Z}$-stable (where $\mathcal{Z}$ denotes the Jiang-Su algebra) if (i) it has finite nuclear dimension or (ii) it is approximately subhomogeneous with slow dimension growth. This generalizes the main results of [27, 36] to the nonunital setting. Algebraic simplicity is established as a fruitful weakening of being simple and unital, and the proof of the main result makes heavy use of this concept.

1. Introduction

The program to classify unital, simple, separable, nuclear C^*-algebras has recently seen a small paradigm shift [9]. In light of the fact that there exist such C^*-algebras that cannot be classified by ordered K-theory and traces [28], the current trend is to try to identify, using regularity properties, the C^*-algebras which are (or should be) classifiable. This idea is crystallized in a conjecture (cf. [30, Section 3.5]), due to Andrew Toms and Wilhelm Winter, that among these C^*-algebras, the following properties are equivalent:

(i) finite nuclear dimension (defined in [38]);
(ii) $\mathcal{Z}$-stability (being isomorphic to one's tensor product with the Jiang-Su algebra $\mathcal{Z}$, introduced in [12]); and
(iii) almost unperforated Cuntz semigroup.

It should be remarked that, at heart, the idea is that, modulo the UCT, conditions (i)-(iii) are equivalent to being classifiable (though in this form, such a statement is not well-formed because classifiability is a property of a class of C^*-algebras, and not a property of a single C^*-algebra).

At the same time, certain examples of simple, stably projectionless C^*-algebras have emerged — including certain crossed products of $\mathcal{O}_2$ by $\mathbb{R}$ [15], a nuclear, separable, non-$\mathcal{Z}$-stable example [25, Theorem 4.1], and others [8, 11, 31]. Certain tools, old and new, already allow one to understand some of the structure of these algebras under special hypotheses [4, 21]. However, most of the theory on regularity properties for C^*-algebras was developed only in the unital case, and leads one to ask what obstructions (if any) exist with nonunital algebras. Certainly, the unital case carries with it a number of simplifications: for instance, the simplex of traces on a C^*-algebra which take the value 1 at the unit (i.e. which are states) plays an indispensible role in the theory, and it is far from obvious what the correct replacement is in the nonunital case. Nonetheless, one hopes that the simplifications in the unital case are superficial, and that ultimately, one can find and prove analogues or generalizations of the known unital results.

The main results of this article are substantiations of this hope. They are a generalization to the nonunital case of the breakthrough results of [36], namely that (i) implies (ii), and that (iii) together with almost divisible Cuntz semigroup and locally finite nuclear dimension imply (ii).

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Algebraic simplicity plays a major role in this article. Simple unital C^*-algebras are automatically algebraically simple (this is a consequence of the fact that elements close enough to the unit are invertible). Algebraically simple algebras therefore extend the class of unital simple algebras, and these algebras seem to retain features that allow certain regularity properties from the unital theory (especially those involving traces) to be phrased in a fruitful way. We show that every simple C^*-algebra is Morita equivalent to an algebraically simple one — and therefore, algebraic simplicity is not itself so much a regularity property, but rather a tool for helping to analyze any C^*-algebra. (Note that the regularity properties appearing in Toms and Winter's conjecture are all preserved under Morita equivalence.)

The result that simple, nuclear C^*-algebras whose Cuntz semigroups are almost unperforated and almost divisible are $\mathcal{Z}$-stable has an important consequence for approximately subhomogeneous C^*-algebras (using a main result of [26]). Namely, it allows us to characterize slow dimension growth for these algebras as equivalent to $\mathcal{Z}$-stability. In the unital case, this result is obtained as a combination of results by Toms [27] and Winter [36]. Many of the motivating examples of stably projectionless C^*-algebras are known to be approximately subhomogeneous [8, 11, 25, 31]. In fact, the class of approximately subhomogeneous, stably projectionless algebras is known to include some crossed products of $\mathcal{O}_2$ by $\mathbb{R}$ [7].

It should be noted that Norio Nawata has generalized in [17, Theorem 5.11] a $\mathcal{Z}$-stability theorem of Matui and Sato [16, Theorem 1.1]. Although they were found independently, many of the Nawata's innovations are similar to those found here.

The organization of this article is as follows. Traces are absolutely indispensible to the results of this article; these are introduced in in Section 2 in connection with algebraically simple C^*-algebras, for whom analysis involving traces is most tractable. Properties closely related to traces also appear in Section 2. We recall the definition of nuclear dimension in Section 3, together with the supporting concept of order zero maps (which reappear in a characterization of $\mathcal{Z}$-stability). Section 4 concerns another extremely useful tool, the asymptotic sequence algebra. This algebra is used throughout the article, largely to compress approximate notions (e.g. elements that approximately commute) and make proofs more conceptual. We recall what the Jiang-Su algebra is, and provide a characterization of when (nonunital) algebras are $\mathcal{Z}$-stable in Section 5 (most of this section actually applies to strongly self-absorbing C^*-algebras in place of $\mathcal{Z}$).

The remaining sections contain the steps of the proofs of the main results. Both main results stated above are reduced to showing that algebras with locally finite nuclear dimension, m-almost divisible Cuntz semigroup and m-comparison (see Definition 2.7) are $\mathcal{Z}$-stable. Particularly, Section 6 shows that finite nuclear dimension implies m-almost divisible Cuntz semigroup for some m, while [22] shows that it implies m-comparison. It is proven in Section 7 that if a C^*-algebra has locally finite nuclear dimension, then m-almost-divisibility implies 0-almost-divisibility, even in an approximately central way, a conclusion that can be summarized by the existence of embeddings of tracially large, central matrix cones into the asymptotic sequence algebra. Section 8 uses the conclusion of Section 7 and m-almost comparison (and, again, locally finite nuclear dimension) to prove $\mathcal{Z}$-stability, using a characterization in terms of central dimension drop embeddings into the asymptotic sequence algebra modulo the annihilator. Finally, it is shown in Section 9 how the main result here combines with a main result of [26] to characterize slow dimension growth in approximately subhomogeneous C^*-algebras.

1.1. **Notation.** The following function will appear many times in the sequel. For $0 \leq \nu < \eta$, we define $g_{\nu,\eta} \in C_0((0,\infty])$ to be the function that is 0 on $[0,\nu]$, 1 on $[\eta,\infty]$ and linear on $[\nu,\eta]$.

2. Algebraic simplicity and traces

We begin by establishing a class of not-necessarily unital C^*-algebras which are amenable to many of the techniques used to study unital C^*-algebras. The simple C^*-algebras in this class are algebraically simple – which at first seems to be a very restrictive condition, and in particular, excludes $A \otimes \mathcal{K}$ whenever A is finite and stably projectionless. However, we show in Corollary 9.1 that if A is simple and σ-unital, then A is isomorphic to an algebraically simple C^*-algebra (in this class). Of course, not all simple C^*-algebra is algebraically simple (for example, $A \otimes \mathcal{K}$ is never simple when A is finite). We will make great use of Pedersen's minimal dense ideal, first studied in [20], which we will call the Pedersen ideal and denote (for a C^*-algebra A) by $\mathrm{Ped}\,(A)$. A good reference on the Pedersen ideal is [19, Section 5.6].

Theorem 2.1. *Let A be a C^*-algebra. The following are equivalent:*

(i) $A = \mathrm{Ped}\,(A)$ *and A is σ-unital;*
(ii) *There exists a strictly positive element in* $\mathrm{Ped}\,(A)$*;*
(iii) *There exists a strictly positive element for A in* $\mathrm{Ped}\,(A \otimes \mathcal{K}) \cap A$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.

(ii) $\Rightarrow$ (i) holds by [19, Proposition 5.6.2], which says that whenever $a \in \mathrm{Ped}\,(A)$, it follows that $\mathrm{her}(a) \subseteq \mathrm{Ped}\,(A)$.

(iii) $\Rightarrow$ (ii): Let $e \in \mathrm{Ped}\,(A \otimes \mathcal{K}) \cap A$ be strictly positive (for A). Since e is full in $A \otimes \mathcal{K}$, it follows by [26, Proposition 3.1] that $\mathrm{Prim}(A \otimes \mathcal{K})$ is compact. But $\mathrm{Prim}(A \otimes \mathcal{K}) = \mathrm{Prim}(A)$, and then it follows by [26, Proposition 3.1] that $\mathrm{Ped}\,(A)$ contains a full element, a.

a^3 is also full, and so, $\mathrm{Ped}\,(A \otimes \mathcal{K})$ is the algebraic ideal of $A \otimes \mathcal{K}$ generated by a^3. By [19, Proposition 5.6.2], $e^{1/3} \in \mathrm{Ped}\,(A \otimes \mathcal{K})$. Therefore, there exist $x_1, \dots, x_k, y_1, \dots, y_k \in A \otimes \mathcal{K}$ such that

$$e^{1/3} = \sum_{i=1}^{k} x_i a^3 y_i,$$

and so

$$e = \sum_{i=1}^{k} \left(e^{1/3} x_i a\right) \, a \, \left(a y_i e^{1/3}\right),$$

where $e^{1/3} x_i a, a e^{1/3} x_i \in A$. Thus, e is contained in the algebraic ideal of A generated by b, which is clearly $\mathrm{Ped}\,(A)$. □

Corollary 2.2. *Let A be a σ-unital simple C^*-algebra. Then there exists a nonzero hereditary subalgebra B of A such that B is algebraically simple. In particular, A is stably isomorphic to B.*

Proof. We merely take $b \in \mathrm{Ped}\,(A)$ nonzero and set $B = \mathrm{her}(b)$. Since $\mathrm{Ped}\,(A) \subset \mathrm{Ped}\,(A \otimes \mathcal{K}) = \mathrm{Ped}\,(B \otimes \mathcal{K})$, it follows from Theorem 2.1 that B is algebraically simple. Finally, Brown's Theorem [3] shows that A is stably isomorphic to B (which we also used in the last sentence). □

Definition 2.3. *We shall use $T(A)$ to denote the set of densely finite (a.k.a. densely defined) traces, as defined in* [19, Definition 5.2.1]*. Every $\tau \in T(A)$ is defined on all of* $\mathrm{Ped}\,(A)$*. For $\tau \in T(A)$, we denote*

$$\|\tau\| := \sup\{\tau(x) : x \in A_+, \|x\| \le 1\} \in [0, \infty];$$

and we set

$$T^1(A) := \{\tau \in T(A) : \|\tau\| = 1\}.$$

For $\tau \in T(A)$ and $a \in A_+$, we set

$$d_\tau(a) := \lim_{n\to\infty} \tau(a^{1/n}).$$

In the unital case, $T^1(A)$ consists of exactly the traces which take the value 1 at the unit. We record the following easily verified generalization to the nonunital setting.

Proposition 2.4. *Let $e \in A_+$ be strictly positive. Then for any $\tau \in T(A)$,*

$$\|\tau\| = d_\tau(e).$$

Proof. We may assume that e is contractive, since rescaling e does not change $d_\tau(e)$.
For $a \in A_+$,

$$\tau(a) = \lim_{n\to\infty} \tau(e^{1/n}ae^{1/n}) \leq \lim_{n\to\infty} \|a\|\tau(e^{2/n}) = \|a\|d_\tau(e);$$

and therefore, $\|\tau\| \leq d_\tau(e)$. On the other hand, $\tau(e^{1/n}) \leq \|\tau\|$ for all n, hence $\|\tau\| \geq d_\tau(e)$. □

Proposition 2.5. *Let A satisfy the equivalent conditions of Theorem 2.1. Then*

(i) *Every densely finite trace on A is bounded;*

(ii) *If $a \in A_+$ is full then*

$$\inf_{\tau\in T^1(A)} \tau(a) > 0.$$

Proof. Let $e \in A_+$ be strictly positive, let $a \in A_+$ be full, and let $\tau \in T(A)$. Since e is in the Pedersen ideal, we have

$$[e] \ll \infty[a]$$

in $\mathcal{C}u(A)$, and therefore, there exists M and ϵ such that

$$[e] \leq M[(a-\epsilon)_+].$$

It follows that

$$\begin{aligned}\|\tau\| &= d_\tau(e)\\ &\leq M d_\tau((a-\epsilon)_+)\\ &\leq \frac{M}{\epsilon}\tau(a).\end{aligned}$$

(i) now follows: since $a \in \mathrm{Ped}\,(A)$, we know that $\tau(a) < \infty$.
(ii) also follows since this shows that

$$\inf_{\tau\in T^1(A)} \tau(a) \geq \frac{\epsilon}{M}.$$

□

We will most often use $T^1(A)$ to access the traces on A. However, another base of $T(A)$ is at times better; it is

$$T_{a\to 1}(A) := \{\tau \in T(A) : \tau(a) = 1\}.$$

Proposition 2.6. [26, Proposition 3.4] *Suppose that $a \in \mathrm{Ped}\,(A)_+$ is full. It follows that $T_{a\mapsto 1}(A)$ is:*

(i) *a base for $T(A)$*

(ii) *compact, in the topology of pointwise converge on the Pedersen ideal of A;*

(iii) *a Choquet simplex.*

We will now consider some comparison and divisibility properties that appear in [36, Definitions 2.1 and 2.5], in the unital case. Some of these have already been phrased to satisfaction in a nonunital setting: namely, m-comparison, as defined in [18, Definition 2.8], and m-almost divisible Cuntz semigroup, as defined in [36, Definition 2.5 (i)]. One finds that the other properties generalize reasonably if we insist that we work with algebraically simple C^*-algebras (which do, indeed, generalize unital simple algebras). For clarity, we restate here even the definitions that are unchanged.

Definition 2.7. *Let A be an algebraically simple C^*-algebra and let $m \in \mathbb{N}$, $\delta > 0$*

(i) *We say that A (or $W(A)$) has m-**comparison** if, given $[x], [y_0], \ldots, [y_m] \in W(A)$, if there exists $k \in \mathbb{N}$ such that*

$$(k+1)[x] \leq k[y_i]$$

for $i = 0, \ldots, m$ then it follows that

$$[x] \leq [y_0] + \cdots + [y_m].$$

(ii) *We say that A has **strong tracial** m-**comparison** if, for any nonzero contractions $a, b \in M_\infty(A)_+$, if*

$$d_\tau(a) < \frac{1}{m+1}\tau(b)$$

for all $\tau \in T^1(A)$ then

$$[a] \leq [b]$$

in $W(A)$.

(iii) *We say that A (or $W(A)$) is m-**almost divisibility** if, for any $[a] \in W(A)$ and any $k \in \mathbb{N}$, there exists $[x] \in W(A)$ such that*

$$k[x] \leq [a] \leq (k+1)(m+1)[x].$$

(iv) *We say that A is **tracially** m-**almost divisible** if for any positive contraction $b \in M_\infty(A)_+, k \in \mathbb{N}, \epsilon > 0$, there exists a c.p.c. order zero map $\psi : M_k \to \mathrm{her}(b)$ such that*

$$\tau(\psi(1_k)) \geq \delta\tau(b) - \epsilon$$

for all $\tau \in T^1(A)$.

Let us establish some relationships between these properties.

Proposition 2.8. *Let A be an algebraically simple C^*-algebra.*

(i) *The following are equivalent.*
   (a) *A has m-comparison;*
   (b) *Given $[x], [y_0], \ldots, [y_m] \in W(A)\backslash\{0\}$, if*

$$d_\tau(x) < d_\tau(y_i)$$

for all $\tau \in T^1(A)$ and for $i = 0, \ldots, m$ then it follows that

$$[x] \leq [y_0] + \cdots + [y_m]; \quad \textit{and}$$

   (c) *Given $[x], [y_0], \ldots, [y_m] \in W(A)\backslash\{0\}$, if there exists $k \in \mathbb{N}$ such that*

$$d_\tau(x) < \tau(y_i)$$

for all $\tau \in T^1(A)$ and for $i = 0, \ldots, m$ then it follows that

$$[x] \leq [y_0] + \cdots + [y_m].$$

(ii) *A has strong tracial m-comparison if and only if, for any nonzero contractions $a, b \in M_\infty(A)_+$, if*
$$d_\tau(a) < \frac{1-\epsilon}{m+1}\tau(b)$$
for all $\tau \in T^1(A)$ then
$$[a] \leq [b]$$
in $W(A)$.

(iii) *If A has stable rank one or finite radius of comparison then $W(A)$ is m-almost divisible if $\mathcal{C}u(A)$ is m-almost divisible.*

(iv) *A is tracially m-almost divisible iff for any $a \in M_\infty(A)_+, k \in \mathbb{N}, \epsilon > 0$, there exist nonzero c.p.c. order zero maps $\psi_0 : M_k \to \mathrm{her}(a)$ such that*
$$\tau(\psi(1_k)) \geq \frac{1-\epsilon}{m+1}\tau(a)$$
for all $\tau \in T(A)$.

(v) *If A is m-almost divisible then A is tracially m-almost divisible.*

(vi) *For any m there exists $\tilde{m}$ (independent of A) such that, if A has m-comparison and is tracially m-almost divisible then A has strong tracial m-comparison.*

Proof. The arguments for (i) (b) $\Leftrightarrow$ (c), (ii), (v), and (vi) are exactly the same as the proofs of Propositions 2.3, 2.4 and 2.7 in [36] respectively, as long as we use $T_{e\mapsto 1}(A)$ in place of $QT(A)$ (particularly for the use of Dini's Theorem), use the result of (iv) in the proof of (v), and use the results of (i), (ii), and (iv) in the proof of (vi). Since the relevant inequalities are linear in the traces τ involved, they hold for all $\tau \in T_{e\to 1}(A)$ iff they hold for all $\tau \in T^1(A)$, iff they hold for all $\tau \in T(A)$.

(i) (a) $\Leftrightarrow$ (b): This follows by [18, Proposition 2.1] (the essential ingredient there being [10, Lemma 4.1]).

(iii) follows from the fact that $W(A)$ is hereditary in $\mathcal{C}u(A)$ under the stated hypotheses, by [1, Theorem 4.4.1] and [5, Theorem 3].

(iv) follows from the observation that, fixing $a \in M_\infty(A)_+$, there exist $r, R \in (0, \infty)$ such that
$$\tau(a) \in (r, R)$$
for all $\tau \in T^1(A)$ (which in turn follows from Proposition 2.5). ☐

We now establish some technical results concerning traces.

Proposition 2.9. *Let $\tau \in T(A)$ be bounded. Then τ extends uniquely to a bounded trace $\tilde{\tau}$ on $\mathcal{M}(A)$ satisfying $\|\tilde{\tau}\| = \|\tau\|$; and for $b \in \mathcal{M}(A)$,*
$$\begin{aligned}\tilde{\tau}(b) &= \sup\{\tau(aba) : a \in A, \|a\| \leq 1\};\\ &= \sup_\lambda \tau(e_\lambda b e_\lambda)\end{aligned}$$
for any approximate identity (e_λ).

Proof. Let us first take (e_λ) to be an increasing approximate unit (which exists by [19, Theorem 1.4.2]), and show that
$$\tilde{\tau}(b) := \sup_\lambda \tau(e_\lambda b e_\lambda)$$
defines a bounded trace on $\mathcal{M}(A)$.

Note that if $\lambda < \lambda'$ then $e_\lambda \leq e_{\lambda'}$ and so
$$\tau\left(e_\lambda b e_\lambda\right) = \tau\left(b^{1/2} e_\lambda^2 b^{1/2}\right) \leq \tau\left(b^{1/2} e_{\lambda'}^2 b^{1/2}\right),$$

and therefore,

$$\tilde{\tau}(b) = \lim_\lambda \tau(e_\lambda b e_\lambda).$$

From this, it is clear that $\tilde{\tau}$ is additive, and therefore also positive. It is also evident that $\tilde{\tau}|_A = \tau$.

To see that it satisfies the trace identity, note that for any $x \in \mathcal{M}(A)$, and any λ, we have

$$\tau(e_\lambda x^* x e_\lambda) = \tau(x^* e_\lambda^2 x) = \tilde{\tau}(x^* e_\lambda^2 x) \leq \tilde{\tau}(x^* x),$$

and therefore, $\tilde{\tau}(x^* x) \leq \tilde{\tau}(x x^*)$. The trace identity follows by symmetry.

Let us now show that, for any (not necessarily even increasing) approximate identity (e_λ), and any $b \in \mathcal{M}(A)$, we have

$$\sup\{\tau(aba) : a \in A_+, \|a\| \leq 1\} = \sup_\lambda \tau(e_\lambda b e_\lambda).$$

The inequality $\geq$ is automatic. Conversely, for any contractive $a \in A_+$ and any $\epsilon > 0$, we can find λ such that

$$e_\lambda a \approx_\epsilon a.$$

It follows that

$$\begin{aligned} \tau(aba) &\approx_{2\epsilon\|\tau\|\|b\|} \tau(a e_\lambda b e_\lambda a) \\ &= \tau\left(b^{1/2} e_\lambda a^2 e_\lambda b^{1/2}\right) \\ &\leq \tau\left(b^{1/2} e_\lambda^2 b^{1/2}\right) \\ &= \tau(a e_\lambda b e_\lambda a). \end{aligned}$$

Since ϵ is arbitrary, this establishes the inequality $\leq$.

In addition to verifying that the different formulae for $\tilde{\tau}$ are equivalent, we can now conclude that

$$b \mapsto \sup\{\tau(aba) : a \in A\}$$

satisfies the trace inequality.

Finally, we show uniqueness. Suppose that τ_0 is another trace on $\mathcal{M}(A)$ such that $\tau_0|_A = \tau$ and $\|\tau_0\| = \|\tau\| = \|\tilde{\tau}\|$. Notice that we clearly have $\tilde{\tau} \leq \tau_0$, and in particular, $\tau_0 - \tilde{\tau}$ is itself a trace, which vanishes on A. Moreover,

$$\|\tau_0 - \tilde{\tau}\| = \tau_0(1) - \tilde{\tau}(1) = \|\tau_0\| - \|\tilde{\tau}\| = 0,$$

and therefore, $\tau_0 = \tilde{\tau}$. □

Proposition 2.10. *Let $a \in A_+$ be a positive contraction and suppose $\tau \in T^1(A)$ satisfies $\tau(a) = 1$. Then*

$$\tau(a^{1/2} x a^{1/2}) = \tau(x)$$

for all $x \in A_+$ and

$$\tau(f(a)) = f(1)$$

for all $f \in C_0((0,1])_+$.

Proof. Define $\tilde{\tau} \in T(\mathcal{M}(A))$ by Proposition 2.9. Then

$$\begin{aligned} \tau(x) - \tau(a^{1/2} x a^{1/2}) &= \tau((1-a)x) \\ &= \tau(x^{1/2}(1-a)x^{1/2}) \\ &\leq \tilde{\tau}(1-a) \\ &= 1 - \tau(a) = 0; \end{aligned}$$

on the other hand, $a^{1/2} x a^{1/2} \leq x$ provides the other inequality, so that

$$\tau(x) - \tau(a^{1/2} x a^{1/2}) = 0.$$

Elementary computation shows that the second statement follows from the first, in the case that f is a polynomial. For arbitrary $f \in C_0((0,1])$, $\tau(f(a)) = f(1)$ follows by approximating by polynomials. □

3. Nuclear dimension and order zero maps

The definition of nuclear dimension rests on the following important notion of order zero completely positive contractive maps. They were first defined in [32, Definition 2.1 (b)] for finite dimensional domains (which is all that is used in the present article). In fact, the definition in [32, Definition 2.1 (b)] is slightly different from, though equivalent to, the more elegant definition we state here.

Definition 3.1. *(*[37, Definition 1.3]*) Let A, B be C^*-algebras and let $\phi : A \to B$. We say that ϕ is a* ***completely positive contractive (c.p.c.) order zero*** *map if it is completely positive, contractive, and preserves orthogonality in the sense that if $a, b \in A_+$ satisfy $ab = 0$ then $\phi(a)\phi(b) = 0$.*

Definition 3.2. *(*[38, Definition 2.1]*) Let A be a C^*-algebra. We say that A has* ***nuclear dimension*** *n if n is the least number for which, given $\epsilon > 0$ and $\mathcal{F} \subset A$ finite, there exists a finite dimensional C^*-algebra F and maps*

$$A \stackrel{\psi}{\longrightarrow} F \stackrel{\phi}{\longrightarrow} A$$

such that:

(i) *for all $a \in \mathcal{F}$, $\|\phi\psi(a) - a\| < \epsilon$;*
(ii) *ψ is c.p.c.; and*
(iii) *F decomposes as*

$$F = F_0 \oplus \cdots \oplus F_n$$

such that $\phi|_{F_i}$ is a c.p.c. order zero map for each i.

We will call such

$$A \stackrel{\psi}{\longrightarrow} F \stackrel{\phi}{\longrightarrow} A \tag{3.1}$$

an $(n+1)$-decomposable c.p. approximation of $\mathcal{F}$ to within ϵ.

(Note that in the literature, (3.1) has been called n-decomposable rather than $(n+1)$-decomposable; I hope that my modified notation is less confusing.)

The following is an extremely useful result, completely revealing the structure of order zero maps from finite dimensional C^*-algebras. (It also holds for order zero maps from arbitrary C^*-algebras [37, Corollary 3.1], though this generalization won't be used here.)

Proposition 3.3. *(*[32, Proposition 3.2]*) Let A, F be C^*-algebras with A finite dimensional, and let $\phi : F \to A$ be a c.p.c. order zero map. Then there exists a $*$-homomorphism $\hat{\phi} : C_0((0,1], F) \to A$ such that*

$$\phi(x) = \hat{\phi}(\mathrm{id}_{(0,1]} \otimes x)$$

for all $x \in F$.

We record a nonunital version of a fundamental fact about nuclear dimension.

Proposition 3.4. *Fix m. Given a finite subset $\mathcal{F} \subset A$ and a tolerance $\epsilon > 0$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that, if $e \in A_+$ is a positive contraction satisfies*

$$ey \approx_\delta y$$

for all $y \in \mathcal{F}$, and

$$(F = F^{(0)} \oplus \cdots \oplus F^{(m)}, \psi, \phi)$$

is a $(m+1)$-decomposable c.p. approximation of $\mathcal{G} \cup \{e\}$ to within δ then, for all $a \in \mathcal{F}$ and $i = 0, \ldots, m$,

(i) $\|[\phi\psi(a), \phi^{(i)}\psi^{(i)}(e)]\| < \epsilon$;
(ii) $a\phi^{(i)}\psi^{(i)}(e) \approx_\epsilon \phi^{(i)}\psi^{(i)}(a)$; *and*

Proof. This follows from the first part of the proof of [38, Proposition 4.3] and some bookkeeping, although a typo should be pointed out that slightly obscures this: Equation (5) of [38] should read

$$\|\phi_\lambda^{(i)}\tilde{\psi}_\lambda^{(i)}(a) - \phi_\lambda^{(i)}\tilde{\psi}_\lambda^{(i)}(e_\lambda)\phi_\lambda\tilde{\psi}(a)\| \overset{\lambda\to\infty}{\longrightarrow} 0.$$

□

4. The asymptotic sequence algebra

Definition 4.1. *Let A be a C^*-algebra. The* ***asymptotic sequence algebra*** *of A is defined as*

$$A_\infty := \prod_{n=1}^\infty A / \Big\{(a_n) \in \prod^\infty A : \lim_{n\to\infty} \|a_n\| = 0\Big\}.$$

We shall use π_∞ to denote the quotient map

$$\prod^\infty A \to A_\infty,$$

and ι to denote the canonical inclusion

$$A \to A_\infty$$

given by $\iota(a) := \pi_\infty(a, a, \dots)$.

Definition 4.2. *Let A be a C^*-algebra. We set denote by $T^1_\infty(A)$ the set of all $\tau \in T(A_\infty)$ for which there exists a sequence $(\tau_n) \subseteq T^1(A)$ and an ultrafilter ω such that*

$$\tau(\pi_\infty(a_n)) = \lim_{n\to\omega} \tau_n(a_n) \tag{4.1}$$

for all $\pi_\infty(a_n) \in A_\infty$.

Remark. Notice that for any sequence $(\tau_n) \subseteq T^1(A)$ and any ultrafilter ω, the right-hand side of (4.1) defines a trace in $T^1_\infty(A)$. Also, one can easily see that $T^1_\infty(A) \subseteq T^1(A)$.

Proposition 4.3. *Let $(a_n) \subset A$ be a bounded sequence of self-adjoint elements and set $a := \pi_\infty(a_n) \in A_\infty$. Let $r \in \mathbb{R}$. Then:*

(i) *$\tau(a) > r$ for all $\tau \in T^1_\infty(A)$ if and only if*
$$\liminf_{n\to\infty} \inf_{\tau\in T^1(A)} \tau(a_n) > r;$$
(ii) *$\tau(a) < r$ for all $\tau \in T^1_\infty(A)$ if and only if*
$$\limsup_{n\to\infty} \sup_{\tau\in T^1(A)} \tau(a_n) < r.$$

Proof. (i): The reverse implication is quite evident. Conversely, suppose that

$$\liminf_{n\to\infty} \inf_{\tau\in T^1(A)} \tau(a_n) \le r.$$

This means that there exists an infinite sequence n_k of indices, together with $\tau_{n_k} \in T^1(A)$ for each k such that

$$\lim_{k\to\infty} \tau_{n_k}(a_{n_k}) \le r.$$

Pick τ_n arbitrarily, for $n \notin \{n_k\}$, and let ω be an ultrafilter containing the set $\{n_k\}$. Then set $\tau := \lim_{n\to\omega} \tau_n(\cdot) \in T^1_\infty(A)$, and we find that

$$\tau(a_n) = \lim_k \tau_{n_k}(a_{n_k}) \le r,$$

as required.

(ii) follows from (i) by replacing a_n by $-a_n$. □

4.1. **The diagonal sequence argument.** The diagonal sequence argument will be used very frequently to allow us to prove the existence of elements in A_∞ satisfying some conditions (exactly), by only verifying that the conditions hold approximately. In order to use the diagonal sequence argument, the rules are as follows:

(a) The conditions need to take the form $f(a) = 0$, where $f : A_\infty \to [0,\infty]$ is a function of the form

$$f(\pi_\infty(a_n)) = \limsup_{n\to\infty} f_n(a_n), \tag{4.2}$$

for some functions f_n; we say that a approximately satisfies the condition (up to a tolerance $\epsilon > 0$) if $f(a) < \epsilon$. (At times, we seek not one but finitely or even countably many elements satisfying joint conditions, and therefore ask for multivariable versions of (4.2))

(b) We may use at most countably many conditions; abusing terminology which is anyhow meant to be informal, we shall also use the name "admissable condition" to refer to a countable collection of such conditions (or a statement that is equivalent to a countable collection of such conditions).

(c) One condition must imply that $\|a\| \leq R$ for (fixed) R.

Here are some examples of admissable conditions that we will use.

(i) Given a separable subspace $X \subset A_\infty$, we have the condition on $a \in A_\infty$ that

$$[a, X] = 0; \tag{4.3}$$

approximately satisfying this condition means, for a finite subset $\mathcal{F} \subset X$ and a tolerance $\epsilon > 0$,

$$\|[a, x]\| < \epsilon$$

for all $x \in \mathcal{F}$. Certainly, we find that (4.3) is equivalent to the countable collection of conditions $f^{(k)}(a) = \|[x_k, a]\|$, where $(x_k) \subset X$ is a dense sequence. It is not hard to see that these $f^{(k)}$'s have the form (4.2).

(ii) Given a separable subspace $X \subset A_\infty$, we have the condition on $a \in A_\infty$ that

$$ax = x, \quad \forall\, x \in X;$$

approximately satisfying this condition means, for a finite subset $\mathcal{F} \subset X$ and a tolerance $\epsilon > 0$,

$$\|ax - x\| < \epsilon$$

for all $x \in \mathcal{F}$.

(iii) Given a separable set $X \subset (A_\infty)_+$ and $\gamma > 0$, we have the condition on $a \in (A_\infty)_+$ that

$$\tau(a^{1/2}xa^{1/2}) \geq \gamma\tau(x) \quad \forall\, x \in X, \tau \in T^1_\infty(A);$$

approximately satisfying this condition means, for a finite subset $\mathcal{F} \subset X$ and a tolerance $\epsilon > 0$,

$$\tau(a^{1/2}xa^{1/2}) \geq \gamma\tau(x) - \epsilon \quad \forall\, x \in \mathcal{F}, \tau \in T^1_\infty(A);$$

or

$$\tau(a^{1/2}xa^{1/2}) \geq (\gamma - \epsilon)\tau(x) \quad \forall\, x \in \mathcal{F}, \tau \in T^1_\infty(A).$$

Again, a sequence of conditions of the form (4.2) can be formed using a dense sequence from X.

(iv) Given $e \in (A_\infty)_+$, we have the condition on $a \in A_\infty$ that
$$ea = a;$$
approximately satisfying this condition means, for a tolerance $\epsilon > 0$,
$$\|ea - a\| < \epsilon.$$

(v) A linear map $M_k \to A_\infty$ may be encoded by its behaviour on a basis of M_k, and therefore we may seek linear maps with certain conditions. Such a condition is that the map is c.p.c. order zero. Although we could form approximate versions of this condition, we don't as it is not needed in the sequel.

(vi) A special case of (v) is, for a single element a, the condition $a \geq 0$.

Remark. We warn that we must be careful not to involve conditions that can't be encoded by a countable set of functions of the form (4.2), even if we verify them exactly instead of approximately. The condition $a \in \mathrm{her}(b)$ is an example of a condition which is not allowed, as the following example shows (we generally employ condition (iv) whenever we wish we could use this condition).

Let A be a separable C^*-algebra which does not have an approximate identity consisting of projections, and let $b \in A_+$ be strictly positive. We shall show that, even though there exists $a \in \overline{bA_\infty b}_+$ satisfying $ab = b$ approximately (this is an instance of (ii)), there does not exist $a \in \overline{bA_\infty b}_+$ satisfying $ab = b$ exactly. Certainly, using an approximate identity, we get $a \in A_+$ satisfying $ab \approx b$. But, if $a \in \overline{bA_\infty b}_+$ satisfies $ab = b$ then it follows that $aa = a$. This means that a is a projection, and can therefore be lifted to a projection in $\prod A$, and since $ab = b$, its lift is an approximate identity, contradicting our hypothesis.

Here is a proof of the diagonal sequence argument. We only state it in the one-variable case, but the k-variable (or countably-many variable) case follows from using $A^{\oplus k}$ (respectively $\oplus_{n=1}^\infty A$) in place of A.

Proposition 4.4. *Let (f_n) be a sequence of functions of the form* (4.2)*, and let $R > 0$. Suppose that we can approximately satisfy these conditions by elements of norm at most R, i.e. for every $\epsilon > 0$ and every N, there exists $a \in A_\infty$ such that $\|a\| \leq R$ and*
$$f_n(a) < \epsilon \quad \forall\, n = 1, \dots, N.$$
Then we can exactly satisfy these conditions, i.e. there exists $a \in A_\infty$ such that $\|a\| \leq R$ and
$$f_n(a) = 0 \quad \forall\, n \in \mathbb{N}.$$

Proof. By hypothesis, let $a_N := \pi_\infty\left((a_i^{(N)})_{i=1}^\infty\right) \in A_\infty$ satisfy $\|a_N\| \leq R$.
$$f_n(a_N) < 1/N \tag{4.4}$$
for $n = 1, \dots, N$. Express
$$f_n(\pi_\infty(x_i)) = \limsup_i f_i^{(n)}(x_i)$$
for some functions $f_i^{(n)}$. Then (4.4) means that there exists i_N such that
$$f_i^{(n)}(a_i^{(N)}) < 1/N$$
and
$$\|a_i^{(N)}\| < R + 1/N$$
for all $n = 1, \dots, N$ and all $i \geq i_N$. We may assume that (i_N) is an increasing sequence. Define
$$a_i = a_i^{(N)}$$

for $i = i_N, \dots, i_{N+1} - 1$. This defines $(a_i) \in \prod A$, and one readily verifies that $\|a\| \leq R$ and

$$f_n(a) = 0$$

for all n. $\square$

4.2. **Almost comparison, almost divisibility, and the asymptotic sequence algebra.**

Proposition 4.5. *Let A be an algebraically simple C^*-algebra. A has strong tracial m-comparison if and only if for any positive contractions $a, b \in M_\infty(A_\infty)_+$, if for all $\epsilon > 0$,*

$$\inf_{\tau \in T^1_\infty(A)} \frac{1}{m+1}\tau(b) - \tau(g_{0,\epsilon}(a)) > 0,$$

then $[a] \leq [b]$ in $W(A)$.

Proof. ($\Rightarrow$): Suppose that A has strong tracial m-comparison and that $a, b \in M_n(A_\infty)_+$ are positive contractions which satisfy, for all $\epsilon > 0$,

$$\inf_{\tau \in T^1_\infty(A)} \frac{1}{m+1}\tau(b) - \tau(g_{0,\epsilon}(a)) > 0.$$

Let $a = \pi_\infty(a_i)$ and $b = \pi_\infty(b_i)$ for some sequences $(a_i), (b_i) \subset M_n(A)_+$ of positive contractions. Let $\epsilon > 0$. Pick some

$$\eta \in \left(0, \inf_{\tau \in T^1_\infty(A)} \frac{1}{m+1}\tau(b) - \tau(g_{0,\epsilon}(a))\right),$$

and we see that

$$\tau(g_{0,\epsilon}(a)) \leq \frac{1}{m+1}\tau((b-\eta)_+) \quad \text{for all } \tau \in T^1_\infty(A).$$

By (the proof of) Proposition 4.3, for all i sufficiently large, we then have

$$d_\tau((a_i - \epsilon)_+) \leq \tau(g_{0,\epsilon}(a_i)) < \frac{1}{m+1}\tau((b_i - \eta)_+) \quad \text{for all } \tau \in T^1_\infty(A),$$

and then by strong tracial m-comparison, $[(a_i - \epsilon)_+] \leq [(b_i - \eta)_+]$ in $W(A)$. [23, Proposition 2.4] shows that there exists $x_i \in M_n(A)$ such that

$$x_i^* x_i = (a_i - \epsilon)_+ \quad \text{and} \quad x_i x_i^* \in \mathrm{her}((b_i - \eta)_+).$$

In particular, $\|x_i\| \leq 1$ and $g_{0,\eta}(b_i)$ acts as a unit on $x_i x_i^*$.

Now, set $x = \pi_\infty(x_i) \in A_\infty$, and note that

$$x^* x = (a - \epsilon)_+ \quad \text{and} \quad g_{0,\eta}(b) x x^* = x x^*.$$

Thus, $[(a-\epsilon)_+] \leq [b]$ in $W(A_\infty)$, and since ϵ is arbitrary, $[a] \leq [b]$.

($\Leftarrow$): Given that A satisfies the latter property in the statement of the proposition, let us show that A has strong tracial m-comparison. Thus, let us take positive contractions $a, b \in M_\infty(A)_+$ such that

$$d_\tau(a) < \frac{1}{m+1}\tau(b)$$

for all $\tau \in T^1(A)$. Let $\epsilon > 0$, and let us show that

$$\inf_{\tau \in T^1(A)} \frac{1}{m+1}\tau(b) - \tau(g_{0,\epsilon}(a)) > 0.$$

Certainly, if we fix a strictly positive element $e \in A_+$ then $\tau \mapsto \frac{1}{m+1}\tau(b) - \tau(g_{0,\epsilon}(a))$ is a continuous function on the compact space $T_{e\mapsto 1}(A)$, and therefore, it has a minimum $r > 0$. This means that

$$\frac{1}{m+1}\tau(b) - \tau(g_{0,\epsilon}(a)) > r\tau(e) \quad \text{for all } \tau \in T(A).$$

Moreover, by Proposition 2.5 (ii), there exists $r' > 0$ such that $\tau(e) > r'$ for all $\tau \in T^1(A)$. Thus,

$$\frac{1}{m+1}\tau(b) - \tau(g_{0,\epsilon}(a)) > rr' \quad \text{for all } \tau \in T^1(A).$$

Since ϵ is arbitrary, we have by hypothesis that $[a] \leq [b]$ in $W(A_\infty)$. But of course this implies that $[a] \leq [b]$ in $W(A)$. □

Proposition 4.6. *Suppose that A is an algebraically simple C^*-algebra. Then A is tracially m-almost divisible if and only if, for any positive contractions $b, e \in (A_\infty)_+$ such that $eb = b$, and for any $k \in \mathbb{N}$ there exists a c.p.c. order zero map*

$$\psi : M_k \to A_\infty$$

such that $e\psi(x) = \psi(x)$ for all $x \in M_k$ and

$$\tau(\psi(1)) \geq \frac{1}{m+1}\tau(d) \quad \text{for all } \tau \in T^1_\infty(A).$$

Proof. The forward implication is a direct application of the diagonal sequence argument from Section 4.1.

Conversely, suppose that A satisfies the latter property and that $b \in (A_\infty)_+$ is a positive contraction, $k \in \mathbb{N}$, and $\epsilon > 0$ are given as in the definition of tracially m-almost divisible. By hypothesis, we may find a c.p.c. order zero map $\psi : M_k \to A_\infty$ such that

$$g_{0,\epsilon/2}(b)\psi(x) = \psi(x)$$

for all $x \in M_k$ and

$$\tau(\psi(1)) \geq \frac{1}{m+1}\tau((b - \epsilon/2)_+ \quad \text{for all } \tau \in T^1_\infty(A).$$

In particular, $\tau(\psi(1)) > \frac{1}{m+1}\tau(b) - \epsilon$ for all $\tau \in T^1_\infty(A)$.

By stability of c.p.c. order zero maps [33, Proposition 1.2.4], we may lift ψ to a sequence of c.p.c. order zero maps $\psi_i : M_k \to A$ satisfying

$$g_{0,\epsilon/2}(b)\psi_i(x) = \psi_i(x) \quad \text{for all } x \in M_k,$$

and therefore, $\psi_i(M_k) \subseteq \mathrm{her}(b)$. Then by (the proof of) Proposition 4.3 (i), for i sufficiently large, we have $\tau(\psi_i(1)) > \frac{1}{m+1}\tau(b) - \epsilon$ for all $\tau \in T^1(A)$, as required. □

5. Characterizing $\mathcal{Z}$-stability

The Jiang-Su algebra $\mathcal{Z}$ was originally defined in [12]; we will recall its characterization in Theorem 5.6. It is a strongly self-absorbing C^*-algebra in following sense.

Definition 5.1. ([29, Definition 1.3 (iv)]) *A C^*-algebra $\mathcal{D}$ is* ***strongly self-absorbing*** *if $\mathcal{D} \neq \mathbb{C}$ is unital and there exists an isomorphism*

$$\mathcal{D} \to \mathcal{D} \otimes_{min} \mathcal{D}$$

which is approximately unitarily equivalent to the first-factor embedding

$$\mathcal{D} \to \mathcal{D} \otimes 1_{\mathcal{D}} \subset \mathcal{D} \otimes_{min} \mathcal{D},$$

(by which we mean the map $d \mapsto d \otimes 1_{\mathcal{D}}$).

In the literature, there are a few characterizations of $\mathcal{Z}$-stability involving suitable embeddings into asymptotic sequence algebras. To this list, we shall add related characterizations, Propositions 5.3 and 5.4, which will be useful in the sequel for proving $\mathcal{Z}$-stability for classes of nonunital C^*-algebras.

For a C^*-algebra A, let us define

$$A^\perp = \{b \in A_\infty : bA = Ab = 0\}.$$

Proposition 5.2. *Let A be a σ-unital C^*-algebra. Then $A^\perp$ is an ideal of $A_\infty \cap A'$ and $(A_\infty \cap A')/A^\perp$ is unital.*

Proof. That $A^\perp$ is an ideal is an easy calculation. The proof of [13, Proposition 1.9 (3)] also shows that $(A_\infty \cap A')/A^\perp$ is unital — namely, if $(e_n) \subset A_+$ is an approximate unit of A then its class in $(A_\infty \cap A')/A^\perp$ is the unit. ☐

Proposition 5.3. *(cf.* [13, Proposition 4.11]*) Let A be a separable C^*-algebra and let $\mathcal{D}$ be strongly self-absorbing. Then A is $\mathcal{D}$-stable if and only if there exists a unital $*$-homomorphism*

$$\mathcal{D} \to (A_\infty \cap A')/A^\perp.$$

Proof. To show the forward implication, we shall describe a unital $*$-homomorphism

$$\mathcal{D} \to ((A \otimes \mathcal{D})_\infty \cap (A \otimes \mathcal{D})')/(A \otimes \mathcal{D})^\perp$$

for any separable C^*-algebra A.

Let $(e_n) \subset A_+$ be an approximate unit and let $\psi_n : \mathcal{D} \to \mathcal{D}$ be an approximately central sequence of unital $*$-homomorphisms.

Define a c.p.c. map

$$\phi_n = e_n \otimes \psi_n(\cdot) : \mathcal{D} \to A \otimes \mathcal{D}.$$

For $a \in A$ and $d_1, d_2 \in \mathcal{D}$,

$$\phi_n(d_1)(a \otimes d_2) = (e_n a) \otimes (\psi(d_1) d_2).$$

From this it is clear that the sequence (ϕ_n) induces a c.p.c. map ϕ from $\mathcal{D}$ to $(A \otimes \mathcal{D})_\infty \cap (A \otimes \mathcal{D})'$. Moreover, for $d_1, d_2, d_3 \in \mathcal{D}$ and $a \in A$,

$$\begin{aligned}\phi_n(d_1)\phi_n(d_2)(a \otimes d_3) - \phi_n(d_1 d_2)(a \otimes d_3) &= (e_n^2 a) \otimes (\psi_n(d_1)\psi_n(d_2) d_3) \\ &\quad - (e_n a) \otimes (\psi_n(d_1)\psi_n(d_2) d_3) \\ &= e_n(e_n a - a) \otimes \psi_n(d_1 d_2) d_3 \\ &\to 0\end{aligned}$$

as $n \to \infty$. Consequently, we see that ϕ in fact induces a $*$-homomorphism

$$\mathcal{D} \to ((A \otimes \mathcal{D})_\infty \cap (A \otimes \mathcal{D})')/(A \otimes \mathcal{D})^\perp;$$

it is straightforward (from the description of the unit in the proof of Proposition 5.2) that this map is unital.

Conversely, given a unital $*$-homomorphism

$$\phi : \mathcal{D} \to (A_\infty \cap A')/A^\perp,$$

we may define a map $\psi : A \otimes \mathcal{D} \to A_\infty$ by

$$\psi(a \otimes d) := ax \tag{5.1}$$

where $x \in A_\infty \cap A'$ is a lift of $\phi(d)$. Note that the definition of $A^\perp$ ensures that the right-hand side of (5.1) is independent of the choice of x, i.e. that ψ is well-defined. Moreover, we see that since ϕ is unital, $\psi(a \otimes 1) = a$ for any $a \in A$. Thus, [29, Theorem 2.3] shows that A is $\mathcal{D}$-stable (the hypothesis in [29, Theorem 2.3] that $\mathcal{D}$ is K_1-injective is automatically satisfied by [35, Remark 3.3]. ☐

Suited to the previous characterization of $\mathcal{D}$-stability is the following characterization, in the case that $\mathcal{D}$ is an inductive limit. (We have in mind $\mathcal{Z}$ as a limit of dimension drop algebras.)

Proposition 5.4. *Suppose that $\mathcal{D}$ is a strongly self-absorbing C^*-algebra that can be expressed as*

$$\mathcal{D} = \overline{\bigcup_{i=1}^{\infty} \mathcal{D}_i},$$

where $\mathcal{D}_i$ is a unital nuclear subalgebra for each i, and the sequence $(\mathcal{D}_i)$ is increasing. For a separable C^*-algebra A, A is $\mathcal{D}$-stable if and only if, for every i there exists a unital $*$-homomorphism

$$\mathcal{D}_i \to (A_\infty \cap A')/A^\perp.$$

Proof. Appealing to Proposition 5.3, we will show that the latter condition is equivalent to the existence of a unital $*$-homomorphism

$$\mathcal{D} \to (A_\infty \cap A')/A^\perp.$$

The forward implication is obvious.

Conversely, suppose that for each i,

$$\phi_i : \mathcal{D}_i \to (A_\infty \cap A')/A^\perp$$

is a $*$-homomorphism. It suffices, by a diagonalization argument (see Section 4.1), to find u.c.p. maps

$$\mathcal{D} \to (A_\infty \cap A')/A^\perp$$

which are (point-norm-)approximately multiplicative. Therefore, let $\mathcal{F} \subset \mathcal{D}$ be a finite subset and $\epsilon > 0$ be our tolerance. By density, WLOG, $\mathcal{F} \subset \bigcup_i \mathcal{D}_i$, which is to say that $\mathcal{F} \subset \mathcal{D}_{i_0}$ for some i_0.

Since $\mathcal{D}_{i_0}$ is nuclear, so is the map ϕ_{i_0}, and therefore, there exists a u.c.p. map ψ which factors through a finite dimensional algebra, and which approximates ϕ_{i_0} sufficiently well so that

$$\psi(x)\psi(y) \approx_\epsilon \psi(xy) \tag{5.2}$$

for all $x, y \in \mathcal{F}$. Since ψ factors through a finite dimensional algebra, the Arveson Extension Theorem implies that it extends to a u.c.p. map (which will also be denoted ψ)

$$\mathcal{D} \to (A_\infty \cap A')/A^\perp.$$

This map is sufficiently approximately multiplicative, by (5.2).

□

We shall now discuss the characterization of the Jiang-Su algebra and a presentation of its building blocks, the dimension drop algebras.

Definition 5.5. *For$p, q \in \mathbb{N}$, we denote*

$$\mathcal{Z}_{p,q} := \{f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1_q \text{ and } f(1) \in 1_p \otimes M_q\}.$$

Such a C^*-algebra is called a ***dimension drop algebra****.*

Theorem 5.6. *([12]) $\mathcal{Z}$ is (up to isomorphism) the unique simple inductive limit of dimension drop algebras which has a unique trace and whose only projections are 0 and 1. In fact, $\mathcal{Z}$ is an inductive limit of dimension drop algebras of the form $\mathcal{Z}_{p-1,p}$.*

We recall the following presentation of $\mathcal{Z}_{p-1,p}$. In the following, a "C^*-algebra generated by a c.p.c. order zero map $\Phi : M_n$" means a C^*-algebra A together with a c.p.c. order zero map $\Phi : M_n \to A$ such that $A = C^*(\Phi(M_n))$.

Proposition 5.7. *(Rørdam-Winter) The dimension drop algebra $\mathcal{Z}_{p-1,p}$ is the universal unital C^*-algebra generated by a c.p.c. order zero map $\Phi : M_p$ together with an element v such that*

$$v^*v = (1 - \Phi(1_p))$$

and

$$v = \Phi(e_{11})v.$$

Proof. This is a reformulation of [24, Proposition 5.1 (iii) $\Leftrightarrow$ (iv)]. $\square$

6. Finite nuclear dimension implies divisibility

The following generalizes [36, Proposition 3.4]

Proposition 6.1. *Let A be a simple, separable, nonelementary C^*-algebra with $\dim_{nuc} A \leq m < \infty$ (or more generally, A may be separable with $\dim_{nuc} A \leq m < \infty$ and such that every quotient of every ideal of A is nonelementary). Let $(e_n) \subseteq A$ be an approximate unit and let $X \subset A_\infty$ be a separable subspace.*

Then for any $k \in \mathbb{N}$, there are c.p.c. order zero maps

$$\psi^{(1)}, \ldots, \psi^{((m+1)^2)} : M_k \oplus M_{k+1} \to A_\infty \cap A' \cap X'$$

such that

$$\sum_{j=1}^{(m+1)^2} \psi^{(j)}(1_k \oplus 1_{k+1}) \geq \pi_\infty(e_n).$$

Proof. The proof of [38, Proposition 4.3] shows that we can find

$$\phi^{(1)}, \ldots, \phi^{((m+1)^2)} : M_k \oplus M_{k+1} \to A_\infty \cap A'$$

such that

$$\sum_{j=1}^{(m+1)^2} \phi^{(j)}(1_k \oplus 1_{k+1}) \geq \pi_\infty(e_n)$$

(note that the condition that every quotient of every ideal of A is nonelementary is equivalent to the condition that no hereditary subalgebra of A has a finite dimensional representation). In the proof of [36, Proposition 3.4], a subsequence argument is used (in the case that $e_n = 1$) to get maps that additionally commute with a separable subset X. The argument works here, as long as (e_n) is increasing. However, given any contractive $\pi_\infty(e_n)$ (and in particular, any approximate unit (e_n)), we can find an increasing approximate unit (f_n) such that $\pi_\infty(f_n)\pi_\infty(e_n) = \pi_\infty(e_n)$. Therefore, $\pi_\infty(e_n) \leq \pi_\infty(f_n)$, and it suffices to run the argument with (f_n) in place of (e_n). $\square$

Proposition 6.2. *Let A be a C^*-algebra and let $B \subseteq A_\infty$ be a separable subalgebra. Suppose that $k, \ell \in \mathbb{N}$ and $\gamma > 0$ are such that, for any separable commutative subalgebra $C \subseteq A_\infty \cap B'$ of nuclear dimension at most 1, there exist c.p.c. order zero maps $\psi_i : M_{\ell_i} \to A_\infty \cap (B \cup C)'$ for $i = 1, \ldots, k$ such that*

$$\tau(\sum_{i=1}^{k} \psi_i(1_{\ell_i})b) \geq \gamma\tau(b)$$

for all $b \in C^*(B \cup C)_+$ and all $\tau \in T^1_\infty(A)$, and such that $\ell_i \in [2k, \ell]$ for each i.

Then for any $p \in \mathbb{N}$, there exist pairwise orthogonal contractions $d_1, \ldots, d_{2^p} \in A_\infty \cap B'$ such that

$$\tau(d_i b) \geq \left(\frac{\gamma}{k\ell}\right)^p \tau(b)$$

for all $b \in B_+$, all $i = 1, \ldots, 2^p$, and all $\tau \in T^1_\infty(A)$.

Proof. We prove this by induction. For the case $p = 0$, we may take $d_1 \in A_\infty$ to be any positive contraction which acts as a unit on B.

For the inductive step, let $p \geq 0$, and we begin with $d_1, \ldots, d_{2^p}$ such that

$$\tau\left(d_i b)\right) \geq \left(\frac{\gamma}{k\ell}\right)^p \tau(b)$$

for all $b \in B_+$ and all $\tau \in T^1_\infty(A)$. We want to construct 2^{p+1} elements.

We set $C = C^*(d_1, \ldots, d_{2^p})$, which is a commutative C^*-algebra of nuclear dimension at most 1, and which commutes with B. Therefore, let $\psi_i : M_{\ell_i} \to A_\infty \cap (B \cup C)'$ be as given in the hypothesis. We will use a diagonal sequence argument (see Section 4.1, and in particular, condition (iii)).

Let $\gamma_0 < \frac{\gamma}{k\ell}$ and set $\eta := \frac{\gamma}{k\ell} - \gamma_0$. Define $a := \sum_i \psi_i(e_{11}) \in A_\infty \cap (B \cup C)'$, and note that

$$\tau(ab) \geq \frac{\gamma}{\ell}\tau(b)$$

for all $b \in C^*(C \cup B)_+$ and all $\tau \in S$ (since $\tau(\psi_i(\cdot)b)$ defines a tracial functional on M_{ℓ_i}).

Let $e \in (A_\infty)_+$ be a positive contraction which acts as a unit on $(B \cup C \cup \{a\})$. Define

$$d_1' := g_{\eta,2\eta}(a)$$

and

$$d_2' := e - g_{0,\eta}(a).$$

Using the fact that e acts as an identity on a, we see that $d_2' \geq 0$ and $d_1'd_2' = 0$. For $b \in C^*(B \cup C)_+$, we have $d_1'b \geq \left(\frac{a}{\|a\|} - \eta\right)b$, so that for $\tau \in S$,

$$\tau(d_1'b) \geq \left(\frac{\gamma}{\ell\|a\|} - \eta\right) = \gamma_0\tau(b). \tag{6.1}$$

Also, $b - d_2'b = (e - d_2')b = g_{0,\eta}(a)b$ so that

$$\begin{aligned}
\tau(b) - \tau(d_2'b) &= \tau(g_{0,\eta}(a)b) \\
&\leq \lim_{n\to\infty} \tau(a^{1/n}b) \\
&\leq \sum_i \lim_{n\to\infty} \tau(\psi_i(e_{11})^{1/n}b) \\
&= \sum_i \lim_{n\to\infty} \tau((\psi_i)^{1/n}(e_{11})b) \\
&= \sum_i \lim_{n\to\infty} \frac{1}{\ell_i}\tau((\psi_i)^{1/n}(1_{\ell_i})b) \\
&\leq \left(\sum_i \frac{1}{2k}\right)\tau(b) \\
&\leq \frac{1}{2}\tau(b).
\end{aligned}$$

In particular,

$$\tau(d_2'b) \geq \frac{1}{2}\tau(b) \geq \gamma_0\tau(b). \tag{6.2}$$

Using commutativity, $(d_i'd_j)_{i=1,2,j=1,\ldots,2^p}$ is a family of 2^{p+1} orthogonal positive contractions inside $A_\infty \cap B'$. By (6.1) and (6.2), each member $d_i'd_j$ of this family satisfies

$$\tau(d_i'd_jb) \geq \gamma_0\tau(d_jb) \geq \eta\left(\frac{\gamma}{k\ell}\right)^p \tau(b),$$

for all $b \in B_+$ and all $\tau \in T^1_\infty(A)$. $\square$

Here is a generalization of [36, Proposition 3.7]

Theorem 6.3. *Given m there exists $\tilde{m}$ such that the following holds. If A is algebraically simple, separable, nonelementary, and with nuclear dimension at most m, then A is tracially $\tilde{m}$-almost divisible.*

Proof. Set
$$p := \lceil 2\log_2(m+1)\rceil,$$
and
$$\tilde{m} := 3^p - 1.$$

Let $b \in A_+$ be a positive contraction and let $\epsilon > 0$. Let $\overline{k} \in \mathbb{N}$ be such that
$$\frac{1}{\overline{k}+1} < \frac{\epsilon}{3} \quad \text{and} \quad k|\overline{k}.$$

Let (e_n) be an approximate unit and use Proposition 6.1 to get c.p.c. order zero maps $\phi^{(r)} : M_{\overline{k}} \oplus M_{\overline{k}+1} \to A_\infty \cap \{b\}'$ be such that
$$\sum_{r=1}^{(m+1)^2} \phi^{(r)}(1_k \oplus 1_{k+1}) \geq \pi_\infty(e_n).$$
We now wish to use Proposition 6.2 with $k = (m+1)^2$, $\ell = 2(m+1)^2+1$, and $\gamma = 1$. Proposition 6.1 immediately verifies the hypotheses of Proposition 6.2. Therefore by Proposition 6.2, we obtain pairwise orthogonal contractions
$$d^{(1)}, \dots, d^{((m+1)^2)} \in A_\infty \cap \{b\}' \cap \left(\bigcup_r \phi^{(r)}(M_{\overline{k}} \oplus M_{\overline{k}+1})\right)'$$
such that
$$\tau(d_i c) \geq 3^{-p}\tau(c) = \frac{1}{\tilde{m}+1}\tau(c) \tag{6.3}$$
for all $c \in C^*\left(\{b\} \cup \bigcup_r \phi^{(r)}(M_{\overline{k}} \oplus M_{\overline{k}+1})\right)$.

Define $\overline{\phi} : M_{\overline{k}} \oplus M_{\overline{k}+1} \to A_\infty \cap \{b\}'$ by
$$\overline{\phi}(x) := \sum_r \phi^{(r)}(x) d^{(r)}.$$

Since this is a sum of c.p.c. maps with orthogonal ranges, it is c.p.c. order zero. Moreover, for $\tau \in T^1_\infty(A)$,
$$\begin{aligned}
\tau(\overline{\phi}(1_{\overline{k}} \oplus 1_{\overline{k}+1})b) &= \sum_r \tau\left(\phi^{(r)}(1_{\overline{k}} \oplus 1_{\overline{k}+1}) b d^{(r)}\right) \\
&\geq^{(6.3)} \frac{1}{\tilde{m}+1} \sum_r \tau\left(\phi^{(r)}(1_{\overline{k}} \oplus 1_{\overline{k}+1}) b\right) \\
&\geq \frac{1}{\tilde{m}+1}\tau(b).
\end{aligned}$$

Since $\overline{\phi}$ is order zero, it follows that
$$\tau(\overline{\phi}(1_{\overline{k}} \oplus 1_{\overline{k}})b) \geq \frac{1}{\tilde{m}+1}\left(1 - \frac{1}{\overline{k}+1}\right)\tau(b) \geq \frac{1}{\tilde{m}+1}\tau b - \frac{\epsilon}{2}$$
for all $\tau \in T^1_\infty(A)$. Define $\phi : M_k \to A_\infty \cap \{b\}'$ by
$$\phi(x) = \overline{\phi}(x \otimes 1_{(\overline{k}/k)} \oplus x \otimes 1_{(\overline{k}/k)}),$$
so that $\tau(\phi(1_k)b) \geq \frac{1}{\tilde{m}+1}\tau(b) - \epsilon/3$ for all $\tau \in T^1_\infty(A)$.

Now, let $(\phi_i) : M_k \to \prod A$ be a c.p.c. order zero lift of ϕ. By Proposition 4.3, it follows that for all i sufficiently large, we have

$$\tau(b - \phi_i(1_k)b) < \epsilon/2 \quad \forall\, \tau \in T^1(A).$$

Also, if i is sufficiently large then $\phi_i(M_k)$ approximately commutes with b. If it commutes well enough, then

$$b^{1/2}\phi(\cdot)b^{1/2}$$

is approximately order zero, and therefore by [33, Proposition 1.2.4], there exists an order zero map $\psi : M_k \to \mathrm{her}(b)$ close enough so that

$$\tau(\psi(1_k)) \geq \tau(b^{1/2}\phi(1_k)b^{1/2}) - \epsilon/2 < \frac{1}{\tilde{m}+1}\tau(b) - \epsilon,$$

as required. □

7. Tracially large, approximately central matrix cones

Lemma 7.1. *Let A be a C^*-algebra and let $B, C \subseteq A$ be subalgebras such that $[B, C] = 0$. Then*

$$\dim_{nuc} C^*(B \cup C) \leq (\dim_{nuc} B + 1)(\dim_{nuc} C + 1) - 1;$$

Proof. Note that $C^*(B \cup C)$ is a quotient of an ideal of $B^\sim \otimes C^\sim$. The result holds by [38, Proposition 2.3 (ii) and (iv), Proposition 2.5, and Remark 2.11]. □

Lemma 7.2. *Let A be a C^*-algebra and let $B \subset A_\infty$ be a separable subalgebra of nuclear dimension at most m. Let $d_0, \dots, d_m, e_0, \dots, e_m \in (A_\infty)_+ \cap B'$ and $\gamma > 0$ be such that*

$$\tau(d_i b) \geq \gamma\tau(b) \quad \forall b \in B_+, i = 0, \dots, m, \tau \in T^1_\infty(A), \tag{7.1}$$

and

$$e_i d_i = d_i \quad \forall i = 0, \dots, m.$$

Let $\delta > 0$, and suppose that either:

(i) *B is commutative and A has tracial $\overline{m}$-almost divisibility, such that $\delta = 1/(\overline{m}+1)$; or*

(ii) *A satisfies: for any $\psi : F \to A_\infty$ c.p.c. order zero and any $d \in (A_\infty \cap \psi(F)')_+$, there exists a c.p.c. order zero map $\phi : M_k \to A_\infty \cap \psi(F)' \cap \{d\}'$ such that*

$$\tau(\phi(1_k)\psi(x)) \geq \delta\tau(\psi(x)), \quad \forall x \in F_+, \tau \in T^1_\infty(A).$$

Then there exist $\Phi_0, \dots, \Phi_m : M_k \to A_\infty \cap B'$ such that

$$\tau\left(\sum_{i=0}^{m} \Phi_i(1_k)b\right) \geq \gamma\delta\tau(b) \tag{7.2}$$

for all $b \in B_+$ and all $\tau \in T^1_\infty(A)$, and

$$e_i\Phi_i(x) = \Phi_i(x)$$

for all $x \in M_k$.

Proof. By the diagonal sequence argument (in Section 4.1), it suffices to arrange approximate commutativity and that the trace inequality only approximately holds. Therefore, let us fix a finite subset $\mathcal{F} \subset B$ and a tolerance $\epsilon > 0$. WLOG, $\mathcal{F}$ contains only positive elements.

(i) Set $\eta < \min\{\epsilon/(m+1), \epsilon/2\}$. Let $B \cong C_0(X)$ for some space X. Let $(x_j^{(i)})_{i=0,\dots,m;j=1,\dots,r} \subset X$ and $(f_j^{(i)})_{i=0,\dots,m;j=1,\dots,r}, (g_j^{(i)})_{i=0,\dots,m;j=1,\dots,r} \subset B_+$ be such that:

$$b \approx_\eta \sum_{i,j} b(x_j^{(i)}) f_j^{(i)}, \tag{7.3}$$

for all $b \in F$;

$$b(x) \approx_\eta b(x_j^{(i)}), \tag{7.4}$$

for all $x \in F$ and $x \in \operatorname{supp} g_j^{(i)}$;

$$g_j^{(i)} f_j^{(i)} = f_j^{(i)},$$

for all i, j; and for each i,

$$f_1^{(i)}, \dots, f_r^{(i)}$$

are pairwise orthogonal.

By Proposition 4.6, let $\Phi_j^{(i)} : M_k \to A_\infty$ be c.p.c. order zero maps satisfying

$$e_i g_j^{(i)} \Phi_j^{(i)}(x) = \Phi_j^{(i)}(x)$$

for all i, j and all $x \in M_k$; and

$$\tau(\Phi_j^{(i)}(1_k)) \geq \gamma \tau(d_i f_j^{(i)}). \tag{7.5}$$

Note that (7.4) implies that

$$zb \approx_\eta b(x_j^{(i)}) z$$

for any contraction $z \in \mathrm{her}(g_j^{(i)})$, and in particular,

$$\Phi_j^{(i)}(x) b \approx_\eta b(x_j^{(i)}) \Phi_j^{(i)}(x) \tag{7.6}$$

for any contraction $x \in M_k$.

Set

$$\Phi^{(i)} := \sum_{j=1}^r \Phi_j^{(i)} : M_k \to A_\infty.$$

Since $\Phi^{(i)}$ is a sum of c.p.c. order zero maps with pairwise orthogonal ranges, it is itself c.p.c. order zero.

Let us compute, for $b \in \mathcal{F}$, and any contraction $x \in M_k$

$$\begin{aligned} \Phi^{(i)}(x) b &= \sum_{j=1}^r \Phi_j^{(i)}(x) b \\ &\approx_\eta b(x_j^{(i)}) \Phi_j^{(i)}(x); \end{aligned} \tag{7.7}$$

for this approximation, we used (7.6) and the fact that the summands are orthogonal. To see that the summands are, in fact, orthogonal, note that since b commutes with $f_j^{(i)}$,

$$b\Phi_j^{(i)}(x) \in \mathrm{her}(g_j^{(i)}) \perp \mathrm{her}(g_{j'}^{(i)}) \quad \text{for } j' \neq j.$$

From (7.7), we obtain first that

$$\left\| \left[b, \Phi^{(i)}(x) \right] \right\| \leq 2\eta < \epsilon.$$

We also find that

$$\begin{aligned}\tau(\Phi^{(i)}(1_k)b) &\geq \sum_{j=1}^{r} b(x_j^{(i)})\tau(\Phi_j^{(i)}(1_k)) - \eta \\ &\geq^{(7.5)} \delta \sum_{j=1}^{r} b(x_j^{(i)})\tau(d_i f_j^{(i)}) - \eta \\ &\geq^{(7.1)} \gamma\delta \sum_{j=1}^{r} b(x_j^{(i)})\tau(f_j^{(i)}) - \eta.\end{aligned}$$

Therefore,

$$\begin{aligned}\tau\left(\sum_{i=0}^{m} \Phi^{(i)}(1_k)b\right) &\geq \gamma\delta\tau\left(\sum_{i,j} b(x_j^{(i)})f_j^{(i)}\right) - m\eta \\ &\geq^{(7.3)} \gamma\delta\tau\left(b\right) - (m+1)\eta,\end{aligned}$$

as required.

(ii) The proof of (ii) is quite similar to that of (i). Set $\eta = \epsilon/(2m+1)$. By Proposition 3.4, let $e \in B_+$ be such that $eb \approx_\eta b$ for $b \in \mathcal{F}$, and let

$$(F = F^{(0)} \oplus \cdots \oplus F^{(m)}, \psi, \phi)$$

be an $(m+1)$-decomposable c.p. approximation for B such that

$$b \approx_\eta \alpha\beta(b)$$

and

$$\alpha_i\beta_i(e)b \approx_{2\eta} \alpha_i\beta_i(b) \tag{7.8}$$

for $b \in \mathcal{F}$ and $i = 0, \ldots, m$.

For each i, let us apply the hypothesis of (ii) to the c.p.c. order zero map $d_i\alpha_i(\cdot)$ and with $d = d_i$, to obtain

$$\hat{\Phi}^{(i)} : M_k \to A_\infty \cap (d_i\alpha_i(F_i))' \cap \{d_i\}' = A_\infty \cap \alpha_i(F_i)' \cap \{d_i\}'$$

such that

$$\tau(\hat{\Phi}^{(i)}(1_k)d_i\alpha_i(x)) \geq \delta\tau(d_i\alpha_i(x)), \quad \forall x \in (F_i)_+, \tau \in T^1_\infty(A). \tag{7.9}$$

Set $\Phi^{(i)} = d_i\alpha_i\beta_i(e)\hat{\Phi}^{(i)}(\cdot)$, which is c.p.c. order zero since $\alpha_i(1_{F_i})$ commutes with the image of $\hat{\Phi}^{(i)}$. Then we have, for any contraction $x \in M_k$ and any $b \in \mathcal{F}$,

$$\begin{aligned}\Phi^{(i)}(x)b &= \hat{\Phi}^{(i)}(x)d_i\alpha_i\beta_i(e)b \\ &\approx_{2\eta}^{(7.8)} \hat{\Phi}^{(i)}(x)\alpha_i(\beta_i(b))\end{aligned}$$

From this we see first that $\|[\Phi^{(i)}(x), b]\| \leq 4\eta$. Secondly, we find that

$$\begin{aligned}\tau\left(\Phi^{(i)}(x)b\right) &\geq \tau(\hat{\Phi}^{(i)}(x)d_i\alpha_i(\beta_i(b))) - 2\eta \\ &\geq_{(7.9)} \delta\tau(d_i\alpha_i(\beta_i(b))) - 2\eta \\ &\geq_{(7.1)} \delta\gamma\tau(\alpha_i\beta_i(b)) - 2\eta.\end{aligned}$$

Summing this over all i gives

$$\begin{aligned}\tau\left(\sum_{i=0}^{m} \Phi^{(i)}(x)b\right) &\geq \delta\gamma\tau\left(\sum_{i=0}^{m} \alpha_i\beta_i(b)\right) - 2m\eta \\ &= \delta\gamma\tau\left(\alpha\beta(b)\right) - 2m\eta \\ &\geq \delta\gamma\tau(b) - (2m+1)\eta,\end{aligned}$$

as required. $\square$

Lemma 7.3. *Let A be a C^*-algebra and let $B \subseteq A_\infty$ be a separable subalgebra of nuclear dimension at most m. Let $\delta > 0$ such that either condition (i) or (ii) of Lemma 7.2 holds. Then for $p \geq 0$, there exist orthogonal contractions $d_1, \dots, d_{2^p} \in A_\infty \cap B'$ such that*

$$\tau(d_i b) \geq \left(\frac{\delta}{(m+2)^2}\right)^p \tau(b)$$

for all $b \in B_+$ and all $\tau \in T^1_\infty(A)$.

Proof. We may find $e, d \in A_\infty$ such that e acts as a unit on d and d acts as a unit on B (since B is separable). We use Lemma 7.2 with $d_i = d$ and $e_i = e$ for all i, then use its conclusion as the hypothesis to Proposition 6.2 to get the conclusion of this lemma. □

Lemma 7.4. *Let A be a C^*-algebra and let $B \subseteq A_\infty$ be a separable subalgebra of nuclear dimension at most m. Let $\delta > 0$ such that either condition (i) or (ii) of Lemma 7.2 holds. Then there exists a c.p.c. order zero map $\phi : M_k \to A_\infty \cap B'$ such that*

$$\tau(\phi(1_k)b) \geq \gamma_{m,\delta}\tau(b) \tag{7.10}$$

for all $b \in B_+$ and all $\tau \in T^1_\infty(A)$, where $\gamma_{m,\delta}$ is the constant

$$\gamma_{m,\delta} := \delta\left(\frac{\delta}{(m+2)^2}\right)^{\lceil \log_2(m+1)\rceil}. \tag{7.11}$$

Proof. By a diagonal sequence argument (see Section 4.1), it suffices to show that (7.10) holds for any $\gamma' < \gamma_{m,\delta}$ in place of $\gamma_{m,\delta}$. Given such a γ', set

$$\eta := \frac{\gamma_{m,\delta} - \gamma'}{\delta} > 0.$$

By Lemma 7.3, there exist orthogonal positive contractions $\overline{d}_0, \dots, \overline{d}_m \in A_\infty \cap B'$ such that

$$\tau(\overline{d}_i b) \geq \gamma_{m,\delta}\tau(b)$$

for all $b \in B_+, i = 0, \dots, m$ and all $\tau \in T^1_\infty(A)$. Let $d_i = (\overline{d}_i - \eta)_+$ and $e_i = g_{0,\eta}(d_i)$, so that $e_i d_i = d_i$. We then have that

$$\begin{aligned}\tau(d_i b) &\geq \tau((\overline{d_i} - \eta)b)\\ &\geq (\gamma_{m,\delta} - \eta)\,\tau(b)\\ &= \frac{\gamma'}{\delta}\tau(b).\end{aligned}$$

for all $b \in B_+, i = 0, \dots, m$ and all $\tau \in T^1_\infty(A)$.

We may then use these with Lemma 7.2, with δ as already provided and with

$$\gamma = \frac{\gamma'}{\delta},$$

to get c.p.c. order zero maps $\phi_i : M_k \to A_\infty \cap B'$ for $i = 0, \dots, m$ such that

$$\tau\left(\textstyle\sum_{i=0}^m \phi_i(1_k)b\right) \geq \gamma'\tau(b)$$

for all $b \in B_+, i = 0, \dots, m$ and all $\tau \in T^1_\infty(A)$, and such that $e_i\phi_i(x) = \phi_i(x) = \phi_i(x)e_i$ for all $x \in M_k$.

Since $e_0, \dots, e_m$ are orthogonal, it follows that $\phi_0, \dots, \phi_m$ have orthogonal ranges. Thus,

$$\phi := \sum_{i=0}^m \phi_i$$

is itself c.p.c. order zero. □

The following builds on an argument appearing in the proof of [36, Lemma 4.11].

Lemma 7.5. *Let A be a separable, algebraically simple C^*-algebra. For a set $\mathcal{B}$ of subalgebras of A_∞ and $k \in \mathbb{N}$, set $\beta_{\mathcal{B},k}$ to be the maximum of $\beta > 0$ such that, for all $B \in \mathcal{B}$, there exists a c.p.c. order zero map $\phi : M_k \to A_\infty \cap B'$ such that*

$$\tau(\phi(1_k)b) \geq \beta\tau(b) \quad \forall b \in B_+. \tag{7.12}$$

Let $\mathcal{B}$ be a set of subalgebras of A_∞ and let $\mathcal{B}'$ consist of all subalgebras of A_∞ of the form $C^*(B \cup C)$ where $B \in \mathcal{B}$, C is the image of an order zero map from a finite dimensional C^*-algebra, and $[B, C] = 0$. Then for $k \in \mathbb{N}$, either

$$\beta_{\mathcal{B}',k}\beta_{\mathcal{B},2} = 0 \quad \text{or} \quad \beta_{\mathcal{B},k} = 1.$$

Proof. First, we note that $\beta_{\mathcal{B},k}$ is truly a maximum, by a diagonal sequence argument (Section 4.1). Suppose that $\beta_{\mathcal{B}',k}\beta_{\mathcal{B},2} \neq 0$.

Set $\eta := 1 - \beta_{\mathcal{B},k} \geq 1 - \beta_{\mathcal{B}',k} > 0$. We shall show that

$$\eta \leq \left(1 - \frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2}\right)\eta + \epsilon, \tag{7.13}$$

for any $\epsilon > 0$, from which it easily follows that $\eta = 0$.

To this end, let $\epsilon > 0$ and let $B \in \mathcal{B}$. Let $\phi : M_k \to A_\infty \cap B'$ be c.p.c. order zero such that (7.12) holds with $\beta = \beta_{\mathcal{B},k}$. $C^*(B \cup \phi(M_k)) \in \mathcal{B}'$, so that there exists $\rho : M_2 \to A_\infty \cap (B \cup \phi(M_k))'$ such that

$$\tau(\rho(1_2)b) \geq \beta_{\mathcal{B}',2}\tau(b)$$

for all $b \in C^*(B \cup \phi(M_k))_+$. Set $d_i = \rho(e_{ii})$ for $i = 1, 2$, so that $d_1, d_2 \in A_\infty \cap (B \cup \phi(M_k))'$ are orthogonal positive contractions satisfying

$$\tau(d_i b) \geq \frac{\beta_{\mathcal{B}',2}}{2}\tau(b) \tag{7.14}$$

for all $b \in C^*(B \cup \phi(M_k))_+$.

Set

$$h_1 := 1 - g_{0,\epsilon}, h_2 := g_{0,\epsilon} - g_{\epsilon,2\epsilon}, h_3 := g_{\epsilon,2\epsilon} \in C([0, 1]), \tag{7.15}$$

which of course satisfy

$$h_1 + h_2 + h_3 = 1. \tag{7.16}$$

Since $C^*(B \cup \{d_1h_1(\phi(1_k))\}) \in \mathcal{B}'$, there exists $\psi : M_k \to A_\infty \cap (B \cup \{d_1h_1(\phi(1_k))\})'$ such that

$$\tau(\psi(1_k)b) \geq \beta_{\mathcal{B}',k}\tau(b), \tag{7.17}$$

for all $b \in C^*(B \cup \{d_1h_1(\phi(1_k))\})_+$.

Now, notice that $d_2h_2(\phi)(\cdot) + h_3(\phi)(\cdot)$ is c.p.c. order zero, since it is c.p. and dominated by $g_{0,\epsilon}(\phi)$. Also, it is orthogonal to the c.p.c. order zero map $d_1h_1(\phi(1_k))\psi(\cdot)$. Thus,

$$\Phi := d_1h_1(\phi(1_k))\psi(\cdot) + d_2h_2(\phi)(\cdot) + h_3(\phi)(\cdot) : M_k \to A_\infty \cap B'$$

is a c.p.c. order zero map. Moreover, for $b \in B_+$ and $\tau \in T^1_\infty(A)$, we have

$$\begin{aligned}
\tau(\Phi(1_k)b) &= \tau(d_1 h_1(\phi(1_k))\psi(1_k)b + d_2 h_2(\phi(1_k))b + h_3(\phi(1_k))b) \\
&\geq^{(7.14),(7.17)} \tau\left(\frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2} h_1(\phi(1_k))b + \frac{\beta_{\mathcal{B}',2}}{2} h_2(\phi(1_k))b + h_3(\phi(1_k))b\right) \\
&\geq^{(7.16)} \frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2}\tau(b) + \left(1 - \frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2}\right)\tau(h_3(\phi(1_k))b) \\
&\geq^{(7.15)} \frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2}\tau(b) + \left(1 - \frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2}\right)\tau((\phi(1_k) - \epsilon)b) \\
&\geq^{(7.17)} \left(\frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2} + \left(1 - \frac{\beta_{2m-1,\tilde{m}^2}}{2}\right)(\beta_{\mathcal{B},k} - \epsilon)\right)\tau(b) \\
&\geq \left(1 - \left(1 - \frac{\beta_{\mathcal{B}',2}\beta_{\mathcal{B}',k}}{2}\right)\eta - \epsilon\right)\tau(b).
\end{aligned}$$

Thus, Φ witnesses (7.13) (for the given B), as required. □

Theorem 7.6. *Let A be a separable, algebraically simple C^*-algebra with tracial m-almost-divisibility. Let $B \subset A_\infty$ a separable subalgebra with nuclear dimension at most $\tilde{m}$, and let $k \in \mathbb{N}$. Then there exists a c.p.c. order zero map $\Phi : M_k \to A_\infty \cap B'$ such that*

$$\tau(\Phi(1_k)b) = \tau(b) \tag{7.18}$$

for all $b \in B_+$ and all $\tau \in T^1_\infty(A)$.

Proof. Let $\mathcal{B}_{\tilde{m}}$ be the set of all separable subalgebras of A_∞ of nuclear dimension at most $\tilde{m}$. Defining $\mathcal{B}'_{\tilde{m}}$ as in Lemma 7.5, we note that by Lemma 7.1, we have $\mathcal{B}'_{\tilde{m}} \subseteq \mathcal{B}_{2\tilde{m}+1}$. Thus, by Lemma 7.5, it suffices to show that

$$\beta_{\mathcal{B}_{\tilde{m}},k} > 0$$

for all k and $\tilde{m}$.

A satisfies the hypothesis of Lemma 7.2 (i), with $\delta = 1/(\tilde{m}+1)$, and therefore Lemma 7.4 shows that:

(∗) For any separable commutative algebra $C \subset A_\infty$ of nuclear dimension at most ℓ, there exists a c.p.c. order zero map $\phi : M_k \to A_\infty \cap C'$ such that

$$\tau(\phi(1_k)c) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c)$$

for all $c \in C_+$ and all $\tau \in T^1_\infty(A)$.

We wish to show that hypothesis (ii) of Lemma 7.2 holds. This hypothesis shall follow (as will be explained) from strengthening (∗), so that instead of being commutative, C is allowed to be of the form

$$C = \bigoplus_{i=1}^{r} C_0(X_i, M_{n_i}) \tag{7.19}$$

(for some spaces X_i).

(As was pointed out to the author by Stuart White, Lemma 7.5 shows that this strengthening already implies that $\gamma_{\ell,1/(\tilde{m}+1)} = 1$, though of course, this also follows once this entire theorem is proven).

Let us first consider the case that C has the form (7.19), except with $r = 1$. Moreover, given $e \in (A_\infty)_+$ satisfying $ec = c$ for all $c \in C$, we will arrange that

$$e\phi(x) = \phi(x)$$

for all $x \in M_k$.

For a finite subset $\mathcal{F} \subset C_0(X_1)_+$ and a tolerance $\epsilon > 0$, we will construct a c.p.c. order zero map $\phi : M_k \to A_\infty \cap C'$ such that $e\phi(x) = \phi(x)$ for all $x \in M_k$ and

$$\tau\left(\phi(1_k)c \otimes a\right) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c \otimes a) - \epsilon,$$

for any $c \in F$. From this, it follows by a diagonal sequence argument (see Section 4.1) that there exists $\phi : M_k \to A_\infty \cap C'$ c.p.c. order zero such that $e\phi(x) = \phi(x)$ for all $x \in M_k$ and

$$\tau\left(\phi(1_k)d\right) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(d),$$

for any $d = \sum_{i=1}^p c_i \otimes a_i$, where $c_1, \ldots, c_p \in C_0(X_1)_+$ and $a_1, \ldots, a_k \in (M_{n_1})_+$; and by continuity, also for any d which can be approximated by elements of such a form.

But, every $d \in C_+$ can be approximated by such elements. Certainly, given $\epsilon > 0$, we may find an open cover $U_1, \ldots, U_n$ of X_1 and points $x_1, \ldots, x_n \in X_1$ such that for $y \in U_i$, $\|d(x_i) - d(y)\| < \epsilon/2$. We may also find positive functions $e_1, \ldots, e_n \in C_0(X_1)$ such that e_i is supported on U_i and

$$\sum e_i d \approx_{\epsilon/2} d.$$

It then follows that

$$d \approx_\epsilon \sum_{i=1}^n e_i \otimes \left(d(x_i)\right).$$

Therefore, let us fix $\mathcal{F} \subset C_0(X_1)_+$ finite and $\epsilon > 0$, and proceed to construct ϕ. $(*)$ provides us with $\psi : M_k \to A_\infty \cap (C_0(X_1) \otimes e_{11})'$ such that

$$\tau(\psi(1_k)c \otimes e_{11}) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c \otimes e_{11}) \tag{7.20}$$

Let $a \in C_0(X_1)$ be a positive contraction such that

$$aca \approx_{\epsilon/n_1} c \tag{7.21}$$

for all $c \in \mathcal{F}$. Then define $\phi : M_k \to A_\infty$ by

$$\phi(x) := \sum_{i=1}^{n_1} (a \otimes e_{i1})\psi(x)(a \otimes e_{1i}).$$

Since e acts as a unit on C, $e\phi(x) = \phi(x)$ for all $x \in M_k$, and since ϕ is defined as a sum of c.p.c. order zero maps with orthogonal images, it is c.p.c. order zero. To see that the image of ϕ commutes with $C = \mathrm{span}\,\{c \otimes e_{ij} : c \in C_0(X_1), i,j = 1, \ldots, n_1\}$, we compute

$$\begin{aligned}
\phi(x)(c \otimes e_{ij}) &= \sum_{k=1}^{n_1} (a \otimes e_{k1})\psi(x)(a \otimes e_{1k})(c \otimes e_{ij}) \\
&= (a \otimes e_{i1})\psi(x)(ca \otimes e_{1j}) \\
&= (a \otimes e_{i1})\psi(x)(c \otimes e_{11})(a \otimes e_{1j}) \\
&= (ac \otimes e_{i1})\psi(x)(a \otimes e_{1j}) \\
&= (c \otimes e_{ij})\phi(x),
\end{aligned}$$

where on the fourth line, we have used the fact that $\psi(x)$ commutes with $c \otimes e_{11}$.

To verify the trace inequality, let $a \in (M_{n_1})_+, c \in \mathcal{F}$, and $\tau \in T_\infty^1(A)$. If $\lambda_1, \ldots, \lambda_{n_1}$ are the eigenvalues of a then there exists $v \in C$ such that

$$v^*v = c \otimes a \quad \text{and} \quad vv^* = c \otimes \left(\sum_{i=1}^{n_1} \lambda_i e_{ii}\right),$$

and thus,

$$\begin{aligned}
\tau\left(\phi(1_k)(c\otimes a)\right) &= \sum_{i=1}^{n_1} \lambda_i \tau\left(\phi(1_k)(c\otimes e_{ii})\right) \\
&= \sum_{i=1}^{n_1} \lambda_i \tau\left(\phi(1_k)(c\otimes e_{11})\right) \\
&= \sum_{i=1}^{n_1} \lambda_i \tau((a\otimes e_{11})\psi(1_k)(a\otimes e_{11})(c\otimes e_{11})) \\
&\geq^{(7.20),(7.21)} \sum_{i=1}^{n_1} \lambda_i \left(\gamma_{\ell,1/(\tilde{m}+1)}\tau(c\otimes e_{11}) - \frac{\epsilon}{n_1}\right) \\
&= \gamma_{\ell,1/(\tilde{m}+1)}\tau(c\otimes a) - \sum_{i=1}^{n_1} \lambda_i \frac{\epsilon}{n_1} \\
&\geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c\otimes a) - \epsilon.
\end{aligned}$$

This concludes the verification of the fact that, for any separable algebra $C \subset A_\infty$ of the form (7.19) with $r = 1$, and of nuclear dimension at most ℓ, and for any $e \in A_\infty$ such that $ec = c$ for all $c \in C$, there exists $\phi : M_k \to A_\infty \cap C'$ such that

$$\tau(\phi(1_k)c) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c)$$

for all $c \in C_+$ and all $\tau \in T^1_\infty(A)$ and

$$e\phi(x) = \phi(x)$$

for all $x \in M_k$.

Now, let $C \subset A_\infty$ be of the form (7.19) with arbitrary r. Once again, fixing a finite subset $\mathcal{F} \subset C_+$ and a tolerance $\epsilon > 0$, we will construct ϕ whose range commutes with $\mathcal{F}$ and satisfies

$$\tau\left(\phi(1_k)c\right) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c) - \epsilon,$$

for any $c \in \mathcal{F}$; and from this, the full result for $C \subset A_\infty$ of the form (7.19).

WLOG by approximating, we may assume that $\mathcal{F}$ consists of compactly supported functions; that is, that $\mathcal{F}$ is contained in

$$\bigoplus_{i=1}^{r} C_0(Y_i, M_{n_i})$$

where Y_i is compactly contained in X_i. Let $e_i \in C_0(X_i)$ be such that $e_i|_{Y_i} \equiv 1$. We have just shown that there exist $\phi_i : M_k \to A_\infty \cap C_0(Y_i, M_{n_i})'$ for each i such that $e_i\phi_i(x) = \phi_i(x)$ for all $x \in M_k$ and

$$\tau(\phi(1_k)c) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c)$$

for all $c \in C_0(Y_i, M_{n_i})_+$ and all $\tau \in T^1_\infty(A)$.

Since e_i acts as a unit on the range of ϕ_i, $\phi_1, \dots, \phi_r$ have orthogonal ranges and therefore $\phi := \sum_{i=1}^n \phi_i$ is c.p.c. order zero. It is also clear that the range of ϕ commutes with $\mathcal{F}$ and satisfies

$$\tau(\phi(1_k)c) \geq \gamma_{\ell,1/(\tilde{m}+1)}\tau(c)$$

for all $c \in \mathcal{F}$ and all $\tau \in T^1_\infty(A)$. This concludes the verification that we can strengthen $(*)$ to the case that C is of the form (7.19).

Let us use this to finally verify hypothesis (ii) of Lemma 7.2, with $\delta = \gamma_{2,1/(\tilde{m}+1)}$. Certainly, given a c.p.c. order zero map $\psi : F \to A_\infty$ and $d \in (A_\infty \cap \psi(F)')_+$, we know that $\psi(F)$ is a quotient of $C_0((0,1], F)$ by Proposition 3.3. Since d commutes with $\psi(F)$, it follows that $C := C^*(\psi(F) \cup \{d\})$ has nuclear dimension at most

2 and is of the form (7.19). We have just shown that there exists $\phi : M_k \to A_\infty \cap \psi(F)' \cap \{d\}'$ such that, in particular,

$$\tau(\phi(1_k)\psi(x)) \geq \gamma_{2,1/(\tilde{m}+1)}\tau(\psi(x))$$

for all $x \in F_+$ and $\tau \in T^1_\infty(A)$.

Applying Lemma 7.4 gives us $\Phi : M_k \to A \cap B'$ witnessing $\beta_{m,\tilde{m}} \geq \gamma_{m,\gamma_{2,1/(\tilde{m}+1)}} > 0$, as required. □

8. Central dimension drop embeddings and proof of the main theorem

Roughly following the arguments of [34, Section 4], we prove here that the conclusion of Theorem 7.6 combined with strong tracial m-comparison (and locally finite nuclear dimension) provide central dimension drop embeddings into the algebra described in Proposition 5.2. In [36, Section 5], the arguments of [34, Section 4] were already adapted to use locally finite nuclear dimension in place of finite decomposition rank. While certain innovations are required here to handle the nonunital case, we have also modified Winter's arguments, making more use of the asymptotic sequence algebra for increased conceptuality.

Lemma 8.1. *(cf.* [34, Proposition 4.2]*) Let A be separable, algebraically simple, with strict tracial m-comparison, and such that $T^1(A) \neq \varnothing$. Let $a \in A_+$ be a positive contraction and let*

$$\Phi : M_n \otimes M_2 \to A_\infty \cap \{a\}'$$

be a c.p.c. order zero map for which $\tau(\Phi(1_{M_n\otimes M_2})) = 1$ for all $\tau \in T^1_\infty(A)$. Let $e \in (A_\infty)_+$ satisfy $ea = a$. Then there exists $v \in A_\infty \cap \{a\}'$ such that

(i) *$ev = ve = v$;*
(ii) *$v^*v = a(1 - \Phi(1_{M_n\otimes M_2}))$; and*
(iii) *$v = \Phi(e_{11} \otimes 1_2)\, v$.*

Proof. Using a diagonal sequence argument (see Section 4.1, it suffices to find v which approximately commutes with a and which approximately satisfies conditions (ii), (iii) (while exactly satisfying (i)). Therefore, let us fix a tolerance ϵ.

Let $\eta < \frac{\epsilon}{4}$. There exist $\bar{t} \in \mathbb{N}$, functions $h^{(i)}_t \in C_0((0,1])_+$ and points $\alpha^{(i)}_t \in (0,1]$ for $i = 1,2$ and $t = 1,\ldots,\bar{t}$ such that for $i = 1,2$, $h^{(i)}_1,\ldots,h^{(i)}_{\bar{t}}$ are pairwise orthogonal and

$$\mathrm{id}_{(0,1]} \approx_\eta \sum_{i=1,2}\sum_{t=1}^{\bar{t}} \alpha^{(i)}_t h^{(i)}_t \text{ and} \tag{8.1}$$

$$f(h^{(i)}_t)\mathrm{id}_{(0,1]} \approx_\eta \alpha^{(i)}_t f\left(h^{(i)}_t\right) \tag{8.2}$$

for any contractive $f \in C_0((0,1])$. Set $a^{(i)}_t := h^{(i)}_t(a)$.

Let $f \in C_0((1-\eta^2,1])_+$ satisfy $f(1) = 1$.

Let $J = \{(i,t) : a^{(i)}_t \neq 0\}$. By Proposition 2.10, we find that

$$\tau\left(a^{(i)}_t(1 - \Phi(1_{M_n\otimes M_2}))\right) = 0.$$

On the other hand, for $(i,t) \in J$, by Proposition 2.5 (ii),

$$\gamma^{(i)}_t := \inf_{\tau\in T^1(A)} \tau\left(a^{(i)}_t\right) > 0;$$

and we have

$$\begin{aligned}\tau\left(\Phi\left(e_{11}\otimes e_{ii}\right)a_t^{(i)}\right) &= \frac{1}{2n}\tau\left(\Phi\left(1_{M_n\otimes M_2}\right)a_t^{(i)}\right)\\ &\geq \frac{1}{2n}\tau\left(a_t^{(i)}\right) > \frac{\gamma_t^{(i)}}{2n},\end{aligned}$$

where the first inequality follows from the fact that Φ is order zero and its image commutes with $a_t^{(i)}$, and the second inequality follows from Proposition 2.10. Thus, by Proposition 4.5 and [23, Proposition 2.4], for $t \in J$, there exists $v_t^{(i)} \in A_\infty$ such that

$$\left(v_t^{(i)}\right)^* v_t^{(i)} = \left(a_t^{(i)}\left(1-\Phi\left(1_{M_n\otimes M_2}\right)\right)-\eta\right)_+ \tag{8.3}$$

and

$$v_t^{(i)}\left(v_t^{(i)}\right)^* \in \mathrm{her}(a_t^{(i)}f\left(\Phi\left(e_{11}\otimes e_{ii}\right)\right)) \tag{8.4}$$

Noting that, since

$$a_t^{(i)}f(\Phi\left(e_{11}\otimes e_{ii}\right)) \perp a_{t'}^{(i')}f\left(\Phi\left(e_{11}\otimes e_{i'i'}\right)\right),$$

whenever $(i,t)\neq(i',t')$, it follows that $v_t^{(i)}\left(v_t^{(i)}\right)^* \perp v_{t'}^{(i')}\left(v_{t'}^{(i')}\right)^*$ and therefore,

$$\left(v_t^{(i)}\right)^* v_{t'}^{(i')} = 0. \tag{8.5}$$

By the same arguments, we find that, if $i = 1$ or 2 and if $t \neq t'$ then since $a, \Phi\left(e_{11}\otimes 1_2\right)$ commute with $a_t^{(i)}, a_{t'}^{(i)}$,

$$\begin{aligned}v_t^{(i)}\left(v_{t'}^{(i)}\right)^* &= \left(v_t^{(i)}a\right)\left(v_{t'}^{(i)}a\right)^* = \left(av_t^{(i)}\right)^*\left(av_{t'}^{(i)}\right)\\ &= \left(\Phi\left(e_{11}\otimes 1_2\right)v_t^{(i)}\right)^*\left(\Phi\left(e_{11}\otimes 1_2\right)v_{t'}^{(i)}\right) = 0.\end{aligned} \tag{8.6}$$

Since $v_t^{(i)} \in \mathrm{her}(a_t^{(i)})$ and by (8.2), we see that

$$v_t^{(i)}a \approx_\eta \alpha_t^{(i)}v_t^{(i)} \approx_\eta v_t^{(i)}a. \tag{8.7}$$

Likewise, since $\mathrm{id}_{(0,1]}f \approx_\eta f$,

$$\Phi\left(e_{11}\otimes 1_2\right)v_t^{(i)}\left(v_t^{(i)}\right)^* \approx_\eta v_t^{(i)}\left(v_t^{(i)}\right)^*,$$

from which it follows that

$$\Phi\left(e_{11}\otimes 1_2\right)v_t^{(i)} \approx_{\eta^{1/2}} v_t^{(i)}. \tag{8.8}$$

Now we may define

$$v := \sum_{i=1,2}\sum_{t=1}^{\bar{t}}\left(\alpha_t^{(i)}\right)^{1/2}v_t^{(i)}.$$

Let us now check that v is as required.

v **approximately commutes with** a: We have

$$\begin{aligned}\left\|va - \sum_{i=1,2}\sum_{t=1}^{\bar{t}}\left(\alpha_t^{(i)}\right)^{3/2}v_t^{(i)}\right\| &\leq \sum_{i=1,2}\left\|\sum_t\left(\alpha_t^{(i)}\right)^{1/2}\left(v_t^{(i)}a-\alpha_t^{(i)}v_t^{(i)}\right)\right\|\\ &= \sum_{i=1,2}\max_t\left\|\left(\alpha_t^{(i)}\right)^{1/2}\left(v_t^{(i)}a-\alpha_t^{(i)}v_t^{(i)}\right)\right\|\\ &\leq^{(8.7)} 2\eta,\end{aligned}$$

where the second line uses the fact that the sums are orthogonal (by (8.5) and (8.6)). Likewise, we obtain that

$$av \approx_{2\eta} \sum_{i=1,2} \sum_{t=1}^{\bar{t}} \left(\alpha_t^{(i)}\right)^{3/2} v_t^{(i)},$$

so that altogether, $\|[a,v]\| \leq 4\eta < \epsilon$.

(i) holds: Evidently, $v_t^{(i)} \in \mathrm{her}(a_t^{(i)})$, and therefore, $v \in \mathrm{her}(a)$. It follows that $ev = ve = v$.

(ii) holds: We compute

$$\begin{aligned} v^*v &= \sum_{i,i',t,t'} \left(v_t^{(i)}\right)^* v_{t'}^{(i')} \\ &=^{(8.5)} \sum_{i,t} \left(v_t^{(i)}\right)^* v_t^{(i)} \\ &=^{(8.3)} \sum_{i,t} \alpha_t^{(i)} \left(a_t^{(i)} \left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right) - \eta\right)_+ \\ &\approx_{2\eta} \sum_{i,t} \alpha_t^{(i)} a_t^{(i)} \left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right) \\ &\approx_{\eta}^{(8.1)} a\left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right), \end{aligned}$$

where the fourth line is achieved by, once again, splitting into two orthogonal sums.

(iii) holds approximately: We compute

$$\begin{aligned} \Phi\left(e_{11} \otimes 1_2\right) v &= \sum_{i,t} \Phi\left(e_{11} \otimes 1_2\right) v_t^{(i)} \\ &\approx_{2\eta} v_t^{(i)}, \end{aligned}$$

by once again splitting into two orthogonal sums (by (8.6)) and using (8.8). □

Lemma 8.2. *(cf.* [34, Proposition 4.3]*) Let A be separable, algebraically simple, with strict tracial m-comparison, and such that $T^1(A) \neq \varnothing$. Let F be a finite dimensional C^*-algebra,*

$$\psi : F \to A$$

be a c.p.c. order zero map and

$$\Phi : M_n \otimes M_2 \to A_\infty \cap \psi(F)'$$

be another c.p.c. order zero map for which $\tau(\Phi\left(1_{M_n \otimes M_2}\right)) = 1$ for all $\tau \in T^1_\infty(A)$. Then there exists $v \in A_\infty \cap \psi(F)'$ such that

(i) *$v^*v = \psi(1)(1 - \Phi\left(1_{M_n \otimes M_2}\right))$; and*
(ii) *$v = \Phi\left(e_{11} \otimes 1_2\right) v$.*

Proof. Again, by the diagonal sequence argument (see Section 4.1), it suffices to find v that only approximately satisfies (i). Therefore let us fix a tolerance ϵ.

Let

$$F = M_{r_1} \oplus \cdots \oplus M_{r_s}$$

and denote $\psi_i = \psi|_{M_i}$. By Lemma 8.1, there exists

$$w_i \in A_\infty \cap \{(\psi_i(e_{11}) - \epsilon)_+\}'$$

such that

(i) $g_{0,\epsilon}(\phi_i(e_{11}))w_i = w_i g_{0,\epsilon}(\phi_i(e_{11})) = w_i$;
(ii) $w_i^* w_i = \left(\psi_i(e_{11}) - \epsilon\right)_+ \left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right)$; and
(iii) $w_i = \Phi\left(e_{11} \otimes 1_2\right) w_i$.

Set

$$v := \sum_{i=1}^{s} \sum_{j=1}^{r_i} g_{0,\epsilon}(\psi_i)(e_{j1}) w_i g_{0,\epsilon}(\psi_i)(e_{1j}).$$

By testing on matrix units, we can see that v commutes with $\psi(F)$. Using the fact that $\psi(F)$ commutes with $\Phi(e_{11} \otimes 1_2)$, one can see that (ii) holds. For (i), we compute

$$\begin{aligned}
v^*v &= \sum_{i,j,i',j'} g_{0,\epsilon}(\psi_i)(e_{j1}) w_i^* g_{0,\epsilon}(\psi_i)(e_{1j}) g_{0,\epsilon}(\psi_{i'})(e_{j'1}) w_{i'} g_{0,\epsilon}(\psi_{i'})(e_{1j'}) \\
&= \sum_{i,j} g_{0,\epsilon}(\psi_i)(e_{j1}) w_i^* g_{0,\epsilon}(\psi_i)(e_{1j}) g_{0,\epsilon}(\psi_i)(e_{j1}) w_i g_{0,\epsilon}(\psi_i)(e_{1j}) \\
&= \sum_{i,j} g_{0,\epsilon}(\psi_i)(e_{j1}) w_i^* g_{0,\epsilon}^2(\psi_i)(e_{11}) w_i g_{0,\epsilon}(\psi_i)(e_{1j}) \\
&= \sum_{i,j} g_{0,\epsilon}(\psi_i)(e_{j1}) w_i^* w_i g_{0,\epsilon}(\psi_i)(e_{1j}) \\
&=^{Lemma 8.1(i)} \sum_{i,j} g_{0,\epsilon}(\psi_i)(e_{j1}) (\psi_i(e_{11}) - \epsilon)_+ \left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right) g_{0,\epsilon}(\psi_i)(e_{1j}) \\
&= \sum_{i,j} (\psi_i(e_{jj}) - \epsilon)_+ \left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right) \\
&= (\psi(1_F) - \epsilon)_+ \left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right) \\
&\approx_\epsilon \psi(1_F) \left(1 - \Phi\left(1_{M_n \otimes M_2}\right)\right).
\end{aligned}$$

□

Proposition 8.3. *(cf.* [34, Proposition 4.4]*) Let A be a simple, separable C^*-algebra and let $B \subseteq A$ be a subalgebra with nuclear dimension at most $m < \infty$. Given a finite subset $\mathcal{F} \subset B$, a positive contraction $h \in C_0((0,1])_+$, and $\delta > 0$, there is a finite subset $\mathcal{G} \subset B$ and $\alpha > 0$ such that the following holds:*

Suppose that $e \in B_+$ is a positive contraction such that

$$ex \approx_\alpha x \approx_\alpha xe$$

for all $x \in \mathcal{F}$, that $(F = F^{(0)} \oplus \cdots \oplus F^{(m)}, \sigma, \rho)$ is an $(m+1)$-decomposable c.p. approximation (for B) of $\mathcal{G} \cup \{e\}$ to within α, and that $v_0, \dots, v_m \in A$ are contractions which satisfy

$$\|[\rho^{(i)}(x), v_i]\| \leq \alpha \|x\| \tag{8.9}$$

for all $x \in F^{(i)}$. Then

$$v := \sum_{i=0}^{m} v_i h(\rho^{(i)} \sigma^{(i)}(e)) \tag{8.10}$$

satisfies

$$\|[v, a]\| < \delta$$

for all $a \in \mathcal{F}$.

Proof. The proof of [36, Proposition 5.4] is easily adapted, with help from Proposition 3.4, to show this. Let us explain.

As in the proof of [36, Proposition 5.4], we may assume that the elements of $\mathcal{F}$ are positive contractions. We pick $\overline{\delta}$ and $\overline{h}$ as in the proof of [36, Proposition 5.4]. Proposition 3.4 gives us $\mathcal{G}$ and $\overline{\alpha}$ such that, given e and an $(m+1)$-decomposable c.p. approximation as in the hypothesis (but using $\overline{\alpha}$ in place of α), we have

$$\|\rho^{(i)} \sigma^{(i)}(e) \rho\sigma(a) - \rho^{(i)} \sigma^{(i)}(a)\| < \frac{\overline{\delta}}{\|\overline{h}\|}, \tag{8.11}$$

for all $a \in \mathcal{F}$. We then take $\alpha := \min\{\overline{\alpha}, \overline{\delta}\}$.

We note that the computation for (176) in the proof of [36, Proposition 5.4] adapts directly to show that

$$\left\| \overline{h}(\rho^{(i)}) \left(\sigma^{(i)}(e) \right) - \sigma^{(i)} \rho^{(i)}(e) \overline{h} \left(\rho^{(i)} \left(1_{F^{(i)}} \right) \right) \right\| < \overline{\delta}.$$

Using this in place of (176) and (8.11) in place of (177), and with e in place of 1_A, the rest of proof of [36, Proposition 5.4] can be used verbatim and shows that

$$\|[v, a]\| < \delta,$$

as required. □

Theorem 8.4. *Let A be a separable, algebraically simple C^*-algebra with strong tracial $\tilde{m}$-comparison and let $B \subseteq A$ be a separable subalgebra with nuclear dimension at most m. Suppose that $\Phi : M_n \otimes M_{m+1} \otimes M_2 \to A_\infty \cap B'$ is a c.p.c. order zero map which satisfies*

$$\tau \left(\Phi \left(1_{M_n \otimes M_{m+1} \otimes M_2} \right) \right) = 1$$

for all $\tau \in T^1_\infty(A)$. Then there exists a positive contraction $e \in (A_\infty)_+ \cap \Phi\left(M_n\right)'$ and $v \in A_\infty \cap B'$ such that e acts as a unit on B and

$$v^* v = e - e\Phi \left(1_{M_n \otimes M_{m+1} \otimes M_2} \right) \quad \text{and} \quad v = \Phi \left(e_{11} \otimes 1_{m+1} \otimes 1_2 \right) v. \tag{8.12}$$

Proof. Appealing to the diagonal sequence argument in Section 4.1, we will show that we can obtain $e \in A_\infty \cap \Phi(M_n)'$ which acts approximately as a unit on B and $v \in A_\infty$ which approximately commutes with B, and such that (8.12) holds approximately. Therefore, let us fix a finite subset $\mathcal{F} \subset B$ and a tolerance $\epsilon > 0$. Let $h, k \in C_0((0,1])_+$ be a contractive, and satisfy

$$h(t) \cdot t^{1/2} = k(t) \quad \text{and} \quad k(t) \approx_{\epsilon/(4(m+1))} t^{1/2}.$$

for all $t \in [0,1]$. Let $\mathcal{G}$ and δ be as given by Proposition 8.3, and WLOG, $\delta < \epsilon/2$. Let $e \in B_+$ be a positive contraction such that $ex \approx_\delta x$ for all $x \in \mathcal{F}$. Let $(F = F^{(0)} \oplus \cdots \oplus F^{(m)}, \sigma, \rho)$ be a $(m+1)$-decomposable c.p. approximation (for B) of $\mathcal{G} \cup \{e\}$ to within δ. By Lemma 8.2, let $v_i \in A_\infty \cap \rho^{(i)}(F^{(i)})$ be such that

(i) $v_i^* v_i = \rho^{(i)}(1_{F_i}) \left(1 - \Phi \left(1_{M_n \otimes M_{m+1} \otimes M_2} \right)\right)$; and
(ii) $v_i = \Phi \left(e_{11} \otimes e_{ii} \otimes 1_2 \right) v_i$.

Denote by $\hat{\rho}^{(i)} : C_0((0,1]) \otimes F^{(i)} \to A$ the $*$-homomorphism associated to $\rho^{(i)}$ as in Proposition 3.3. Then

$$\begin{aligned} h(\rho^{(i)}) \sigma^{(i)}(e) \rho^{(i)}(1_{F_i})^{1/2} &= \hat{\rho}^{(i)} \left((h(\mathrm{id}_{(0,1]} \mathrm{id}^{1/2}_{(0,1]}) \otimes \sigma^{(i)}(e) \right) \\ &= \hat{\rho}^{(i)} \left(k(\mathrm{id}_{(0,1]} \otimes \sigma^{(i)}(e) \right) \\ &= k(\rho^{(i)} \sigma^{(i)}(e)) \\ &\approx_{\epsilon/(4(m+1))} \rho^{(i)} \sigma^{(i)}(e)^{1/2} \end{aligned} \tag{8.13}$$

Now, define v as in (8.10). That $\|[v, a]\| \leq \epsilon$ for $a \in \mathcal{F}$ is ensured by Proposition 8.3. It is also clear from Lemma 8.2 (ii) that

$$v\Phi \left(e_{11} \otimes 1_{m+1} \otimes 1_2 \right) = v \tag{8.14}$$

holds exactly. Finally,

$$\begin{aligned}
v^*v &=^{(8.10)} \sum_{i,j} h(\rho^{(i)}\sigma^{(i)}(e))v_i^*v_j h(\rho^{(j)}\sigma^{(j)}(e)) \\
&=^{(8.14)} \sum_{i,j} h(\rho^{(i)}\sigma^{(i)}(e))v_i^*\Phi\left(e_{11}\otimes e_{ii}\otimes 1_2\right)\Phi\left(e_{11}\otimes e_{jj}\otimes 1_2\right)v_j h(\rho^{(j)}\sigma^{(j)}(e)) \\
&=^{Lemma 8.2(i)} \sum_{i} h(\rho^{(i)}\sigma^{(i)}(e))v_i^*v_i h(\rho^{(i)}\sigma^{(i)}(e)) \\
&= \sum_i h(\rho^{(i)}\sigma^{(i)}(e))\rho^{(i)}(1_{F_i})^{1/2}\left(1-\Phi\left(1_{M_n\otimes M_{m+1}\otimes M_2}\right)\right)\rho^{(i)}(1_{F_i})^{1/2}h(\rho^{(i)}\sigma^{(i)}(e)) \\
&\approx^{(8.13)}_{\epsilon/2} \sum_i (\rho^{(i)}\sigma^{(i)}(e))^{1/2}\left(1-\Phi\left(1_{M_n\otimes M_{m+1}\otimes M_2}\right)\right)(\rho^{(i)}\sigma^{(i)}(e))^{1/2} \\
&= \rho\sigma(e)\left(1-\Phi\left(1_{M_n\otimes M_{m+1}\otimes M_2}\right)\right) \\
&\approx_\delta e\left(1-\Phi\left(1_{M_n\otimes M_{m+1}\otimes M_2}\right)\right),
\end{aligned}$$

as required. □

Theorem 8.5. *Let A be a separable, algebraically simple C^*-algebra. Suppose that one of the following hold.*

(i) *A has finite nuclear dimension;*
(ii) *A has locally finite nuclear dimension, strong tracial m-comparison and tracial m-almost divisibility for some m;*
(iii) *A has locally finite nuclear dimension, m-comparison and tracial m-almost divisibility for some m;*
(iv) *A has locally finite nuclear dimension, m-comparison and m-almost divisibility for some m.*

Then A is $\mathcal{Z}$-stable.

Proof. By Theorem 6.3 and [22], (i) ⇒ (iii). By Proposition 2.8, each of (iii) and (iv) imply (ii). Therefore, we shall show $\mathcal{Z}$-stability using (ii).

This must be proven by considering seperately two cases (the first case is well-known but warrants restating for completeness).

In the case that $T^1(A)=\varnothing$, by [2, Theorem 1.2], $A\otimes\mathcal{K}$ contains a nonzero projection p. Set $B:=p(A\otimes\mathcal{K})p$. By strong tracial m-comparison, and since $T^1(A)=\varnothing$, we have $[a]\leq[b]$ for any nonzero $a,b\in M_\infty(B)_+$. In particular, B is purely infinite (see [6]), and therefore by [14, Theorem 3.15], B is $\mathcal{O}_\infty$-stable. But $\mathcal{O}_\infty$-stability is preserved within stable-isomorphism classes, and therefore A is also $\mathcal{O}_\infty$-stable. Finally, since $\mathcal{O}_\infty$ is $\mathcal{Z}$-stable, so is A.

Now we turn to the case where $T^1(A)\neq\varnothing$. In light of Proposition 5.4, it suffices to find, for any n, a unital $*$-homomorphism

$$\mathcal{Z}_{n-1,n}\to(A_\infty\cap A')/A^\perp.$$

By Proposition 5.7, this is the same as finding a c.p.c. order zero map

$$\Phi: M_n\to(A_\infty\cap A')/A^\perp$$

together with an element $v\in(A_\infty\cap A')/A^\perp$ such that

$$v^*v=1-\Phi\left(1_n\right)\quad\text{and}\quad v=\Phi\left(e_{11}\right)v.$$

By the diagonal sequence argument (see Section 4.1), it suffices to do this approximately, finding for a given finite subset $\mathcal{F}\subset A$ and a tolerance $\epsilon>0$ a c.p.c. order zero map $\Phi: M_n\to A_\infty$ and $v\in A_\infty$ such that:

(i) $\|[\Phi(x),a]\|\leq\epsilon\|x\|$ for all $x\in M_n$ and all $a\in\mathcal{F}$;
(ii) $\|[v,a]\|\leq\epsilon$ for all $v\in M_n$ all $a\in\mathcal{F}$;

(iii) $\|(v^*v - (1-\Phi(1)))\,a\| \leq \epsilon$ for all $a \in \mathcal{F}$; and
(iv) $v = \Phi(e_{11})v$.

Therefore, let us be given $\mathcal{F} \subset A$ finite and $\epsilon > 0$. Since A has locally finite nuclear dimension, we may find $B \subseteq A$ such that $\dim_{nuc} B \leq m < \infty$ and for every $a \in \mathcal{F}$, there exists $a' \in B$ such that $a \approx_{\epsilon/2} a'$. By Theorem 7.6, we may find a c.p.c. order zero map $\Phi : M_n \otimes M_{m+1} \otimes M_2 \to A_\infty \cap B'$ such that

$$\tau\left(\Phi\left(1_{M_n \otimes M_{m+1} \otimes M_2}\right)\right) = 1$$

for all $\tau \in T^1_\infty(A)$. By Theorem 8.4, it follows that there exists $e \in (A_\infty)_+ \cap \overline{\Phi}(M_n)'$ and $v \in A_\infty \cap B'$ such that e acts as a unit on B and

$$v^*v = e - e\overline{\Phi}\left(1_{M_n \otimes M_{m+1} \otimes M_2}\right) \quad \text{and} \quad v = \overline{\Phi}(e_{11} \otimes 1 \otimes 1)v.$$

Set $\Phi := \overline{\Phi}\left(\,\cdot \otimes 1_{m+1} \otimes 1_2\right) : M_n \to A_\infty$. Let us verify that this Φ and v approximately satisfies the relations (i.e. check (i)-(iv)). The idea is, that if $\mathcal{F} \subset B$ exactly then they would exactly satisfy the relations.

(i) and (ii) hold: For a contraction $x \in M_n$ and any $a \in \mathcal{F}$, we have

$$\Phi(x)a \approx_{\epsilon/2} \Phi(x)a' = a'\Phi(x) \approx_{\epsilon/2} a\Phi(x);$$

and likewise we can prove that v approximately commutes with $\mathcal{F}$.

(iii) holds: We have

$$\begin{aligned}\|(v^*v - (1-\Phi(1)))\,a\| &\leq \|(v^*v - (1-\Phi(1)))\,a'\| + 2\frac{\epsilon}{2} \\ &\leq \left\|\big((1-\Phi(1))\,e - (1-\Phi(1))\big)a'\right\| + \epsilon \\ &= \epsilon,\end{aligned}$$

since e acts as a unit on B.

Finally, that (iv) holds is quite clear from the choice of Φ. ☐

Corollary 8.6. *Let A be a separable, simple, nonelementary C^*-algebra with finite nuclear dimension. Then A is $\mathcal{Z}$-stable.*

Proof. This is a consequence of Theorem 8.5, Corollary 2.2, and the fact that $\mathcal{Z}$-stability and the value of nuclear dimension are constant under stable isomorphism classes. ☐

Corollary 8.7. *Let A be a separable, simple C^*-algebra with locally finite nuclear dimension. The following are equivalent.*

(i) $A \cong A \otimes \mathcal{Z}$;
(ii) $W(A) \cong W(A \otimes \mathcal{Z})$;
(iii) $\mathcal{C}u(A) \cong \mathcal{C}u(A \otimes \mathcal{Z})$.

Proof. (i) $\Rightarrow$ (ii) and (iii) is obvious.

(ii) $\Rightarrow$ (i): The proof of [36, Proposition 2.7] does not at all use the fact that A is unital, and thereby shows that $W(A \otimes \mathcal{Z})$ has 0-comparison and 0-almost divisible Cuntz semigroup (properties which, we may recall, do not require algebraic simplicity).

By Corollary 2.2, let B be a nonzero hereditary subalgebra of A which is algebraically simple. Then $W(B) \subset W(A) \subset \mathcal{C}u(A) = \mathcal{C}u(B)$. Since $W(A)$ has 0-comparison, so does $W(B)$. It follows from [1, Theorem 4.4.1] that $W(B)$ is hereditary in $\mathcal{C}u(B)$, and therefore also in $W(A)$. Thus, 0-almost divisibility for $W(A)$ implies 0-almost divisibility for $W(B)$. Theorem 8.5 then shows that $A \cong A \otimes \mathcal{Z}$.

(iii) $\Rightarrow$ (i): $\mathcal{C}u(A) \cong W(A \otimes \mathcal{K})$, so by (ii) $\Rightarrow$ (i), we see that (iii) implies that $A \otimes \mathcal{K}$ is $\mathcal{Z}$-stable. Since $\mathcal{Z}$-stability passes to hereditary subalgebras, $A \cong A \otimes \mathcal{Z}$. ☐

9. Approximately subhomogeneous C^*-algebras

An important consequence of Theorem 8.5 is a characterization of slow dimension growth for simple approximately subhomogeneous C^*-algebras. Roughly, slow dimension growth indicates that the algebra has a system for which the ratio of the topological to matricial dimension vanishes in the limit. We refer to [25, Definition 5.3] for a precise definition (which is trickier to produce in the nonunital, as opposed to unital, case). In the unital case, the characterization was shown in [36, Corollary 6.5] and [27, Corollary 1.3].

Lemma 9.1. *Let A be a simple approximately subhomogeneous algebra. Then there exists a nonzero hereditary subalgebra B of A such that B is algebraically simple and approximately subhomogeneous. In particular, A is stably isomorphic to B.*

Moreover, if A has slow dimension growth as in [25, Definition 5.3] *then B can be chosen to also have slow dimension growth.*

Proof. We shall produce A quite as in the proof of Corollary 2.2, except that the element $b \in \mathrm{Ped}\,(A)$ is chosen with some care, to ensure that B is approximately subhomogeneous. Let A be the closed union of an increasing sequence of subhomogeneous subalgebras,

$$A_1 \subseteq A_2 \subseteq \cdots .$$

Now, take $b \in \mathrm{Ped}\,(A_1) \subseteq \mathrm{Ped}\,(A)$. As in the proof of Corollary 2.2, A is stably isomorphic to $B := \overline{bAb}$, which is algebraically simple. Moreover,

$$B = \overline{bAb} = \overline{\bigcup \overline{bA_ib}},$$

and since A_i is subhomogeneous, so is $\overline{bA_ib}$, for each i.

If A has slow dimension growth then we may pick the inductive sequence

$$A_1 \subseteq A_2 \subseteq \cdots ,$$

to witness this, and then it follows (cf. [25, Proposition 5.2]) that

$$\overline{bA_1b} \subseteq \overline{bA_2b} \subseteq \cdots$$

witnesses slow dimension growth for B. □

Corollary 9.2. *Let A be a simple approximately subhomogeneous algebra. Then A has slow dimension growth, as defined in* [25, Definition 5.3]*, if and only if A is $\mathcal{Z}$-stable.*

Proof. ($\Rightarrow$): By Lemma 2.2, we may assume WLOG that A is algebraically simple. By [25, Corollary 5.9], $\mathcal{C}u(A)$ has 0-comparison; since there is an order embedding $W(A) \subset \mathcal{C}u(A)$, it follows that $W(A)$ also has 0-comparison. By [26, Corollary 7.2], $\mathcal{C}u(A)$ has 0-almost divisibility; then by Proposition 2.8 (iii), it follows that A has 0-almost divisibility.

Therefore, B satisfies hypothesis (iv) of Theorem 8.5, whence B is $\mathcal{Z}$-stable.

($\Leftarrow$): Suppose $A \cong A \otimes \mathcal{Z}$. By [25, Proposition 3.6], there exists an inductive system

$$A_1 \to A_2 \to \cdots$$

whose limit is A, such that A_i is recursive subhomogeneous with finite dimensional total space and compact spectrum, and the connecting maps are injective and full. On the other hand, $\mathcal{Z}$ is the inductive limit of a system

$$\mathcal{Z}_{p_1-1,p_1} \to \mathcal{Z}_{p_2-1,p_2} \to \cdots ,$$

for some $p_i \to \infty$. In fact, by taking a tail, we may make p_i as large as we want, so that the ratio of the dimension of the total space of A_i to p_i goes to 0 as $i \to \infty$. Then it follows that

$$A_1 \otimes \mathcal{Z}_{p_1-1,p_1} \to A_2 \otimes \mathcal{Z}_{p_2-1,p_2} \to \cdots$$

is a system whose limit is $A \otimes \mathcal{Z}$, and witnesses that this algebra has slow dimension growth. $\square$

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