Fixed points of ternary involutions and applications

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Joint work with Les Bunce

Ternary rings of operators (TROs)

Definition

A TRO is a norm closed linear subspace $T \subseteq \mathcal{B}(\mathcal{H})$ such that

$$x, y, z \in T \Rightarrow [x, y, z] := xy^*z \in T$$

Examples

$T = A$. $\mathbb{M}_n(T)$. $T = \mathbb{M}_{n,m}(\mathbb{C})$. $T = pAq$. $T^{\mathrm{op}} \subseteq \mathcal{B}(\mathcal{H})^{\mathrm{op}}$.

Notation

$\mathscr{L}_T = \overline{\mathrm{span}\{xy^* : x, y \in T\}}$ $\qquad$ $\mathscr{R}_T = \overline{\mathrm{span}\{y^*z : y, z \in T\}}$

Linking C^*-algebra of T:

$$\mathfrak{L}_T \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathscr{L}_T & T \\ T^* & \mathscr{R}_T \end{pmatrix} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

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Ideals

Definition

$I \subseteq T$ is an *ideal* if it is a norm closed linear subspace with

$$[I, T, T] + [T, I, T] + [T, T, I] \subseteq I$$

Since $x \in I \Rightarrow x = [y, y, y]$ for some $y \in I$, can omit $[T, I, T]$ (or require only $[T, I, T] \subseteq I$).

Proposition

$I \subseteq T$ an ideal implies $\mathscr{R}_I \subseteq \mathscr{R}_T$ an ideal (and so is $\mathscr{L}_I \subseteq \mathscr{L}_T$). Moreover

$$I = T\mathscr{R}_I = \mathscr{L}_I T$$

and $J \subseteq \mathscr{R}_T$ an ideal implies $I_J = TJ \subseteq T$ an ideal with $\mathscr{R}_{I_J} = J$.

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Ternary morphisms (of TROs)

Definition

$\phi \colon T_1 \to T_2$ is a *ternary homomorphism* if
$\phi[x, y, z] = [\phi(x), \phi(y), \phi(z)]$ (or $\phi(xy^*z) = \phi(x)(\phi(y))^*\phi(z)$).

Proposition

Ternary homomorphisms are (completely) contractive.
$\phi \colon T_1 \to T_2$ induces *-homomoprhisms $\mathscr{L}_\phi \colon \mathscr{L}_{T_1} \to \mathscr{L}_{T_2}$ $(xy^* \mapsto \phi(x)(\phi(y))^*)$ and $\mathscr{R}_\phi \colon \mathscr{R}_{T_1} \to \mathscr{R}_{T_2}$ and

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$\rightsquigarrow$ Abstract TRO: $(T, [\cdot, \cdot, \cdot])$.

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Corners & tripotents

$$\mathfrak{L}_T{}^\sim \stackrel{\text{def}}{=} \begin{pmatrix} \mathscr{L}_T{}^\sim & T \\ T^* & \mathscr{R}_T{}^\sim \end{pmatrix} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

Ex

For $p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{L}_T{}^\sim$, $q = 1 - p = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$,

$T \cong \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} = p(\mathfrak{L}_T)q$

Definition

$e \in T$ is called a tripotent if $[e, e, e] = ee^*e = e$ ($\iff$ e a partial isometry)

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Weak*-closed TROs and biduals

If we consider TROs $U \in \mathcal{B}(\mathcal{H})$ that are weak*-closed (or are Banach dual spaces), all extreme points of the unit ball are tripotents.

Bidual or weak* closure of T can be U.

Use $\overline{\mathscr{L}_U} = \overline{\operatorname{span}\{xy^* : x, y \in U\}}^{w^*}$, $\overline{\mathscr{R}_U}$ and

$$\begin{pmatrix} \overline{\mathscr{L}_U} & U \\ U^* & \overline{\mathscr{R}_U} \end{pmatrix} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

Proposition (Zettl)

Weak*-closed 'ideals' $I \subseteq U$ are in 1-1 correspondence with projections $z \in Z(\overline{\mathscr{R}_U})$ via $I = Uz$.

Definition

A W^*-TRO U is called a *left TRO* if U is TRO isomorphic to Wp for $p = p^* = p^2 \in W$, W a W^*-algebra.
U is called *square* if $U \cong W$.
U *square-free* if $\nexists I \subseteq U$ with $I \neq \{0\}$ square.

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Weak*-closed 'ideals' $I \subseteq U$ are in 1-1 correspondence with projections $z \in Z(\overline{\mathscr{R}_U})$ via $I = Uz$.

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Weak*-closed TROs and biduals

If we consider TROs $U \in \mathcal{B}(\mathcal{H})$ that are weak*-closed (or are Banach dual spaces), all extreme points of the unit ball are tripotents.

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Left/right/square decomposition

Theorem

U a W^*-TRO implies

$$U = U_{\mathrm{l}} \oplus U_{\mathrm{r}} \oplus U_{\mathrm{s}}$$

with U_{l}/ U_{r}/ U_{s} the largest square-free left/ square-free right/ square weak-closed ideals of U.*

Example

For $p \in \mathcal{B}(\mathcal{H})$ a projection ($p \neq 0$), $U = \mathcal{B}(\mathcal{H})p$ is a left TRO, $\overline{\mathscr{L}_U} = \mathcal{B}(\mathcal{H})$, no non-trivial (weak*-closed) ideals, square-free if $\dim p(\mathcal{H}) < \dim \mathcal{H}$.
For p rank one, $U = \mathcal{B}(\mathcal{H})p$ is a left TRO, isometric to $\mathcal{H}$ as a Banach space, square-free if $\dim \mathcal{H} > 1$. (Column Hilbert space.)

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An *involution* of a C^*-algebra A is $\Phi\colon A \to A$ such that Φ is $\mathbb{C}$-linear, $\Phi(\Phi(a)) = a$, $\Phi(ab) = \Phi(b)\Phi(a)$, and $\Phi(a^*) = \Phi(a)^*$

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T^ϕ will be a JC^*-triple: closed under Jordan triple product

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A JC^*-triple is a closed linear $E \subseteq \mathcal{B}(\mathcal{H})$ such that

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Examples

$E = T$ or $E = T^\phi$ (*e.g.* with $T = \mathbb{M}_n(\mathbb{C})$, $\phi(x) = x^t$ or $\phi(x) = -x^t$).

We consider 'concrete' JC^*-triples E and F the 'same' if $\exists$ Jordan triple isomorphism $\psi \colon E \to F$ ($\iff$ ψ an isometry).

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On JC^*-triples

Recall: A JC^*-triple is a closed linear $E \subseteq \mathcal{B}(\mathcal{H})$ such that

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Relate to isometric theory of Banach spaces (since triple homomorphisms $\pi\colon E \to F \subseteq \mathcal{B}(\mathcal{K})$ are contractive).

$(E, \{\cdot, \cdot, \cdot\})$ abstract triple has no canonical op. space structure.

Neal & Russo found that for many E, there are only a few.

Example

TROs T give rise to at least 3 obvious concrete JC^*-triples: $E = T$, $E = T^{\mathrm{op}}$ and $E = \{x \oplus x^{\mathrm{op}} : x \in T\} \subseteq T \oplus T^{\mathrm{op}}$

These examples are reversible. In latter case $E = (T \oplus T^{\mathrm{op}})^{\phi}$ where $\phi(x \oplus y^{\mathrm{op}}) = y \oplus x^{\mathrm{op}}$.

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Universal property of $T^*(E)$

Theorem (Bunce, Feely, T (Math. Zeit. 2011))

For each JC^*-triple E there is a largest TRO $T^*(E)$ generated by (triple isomorphic copies of) E

$$\begin{tikzcd} T^*(E) \arrow[dr, dashed, "\tilde{\pi}"] & \\ E \arrow[u, "\alpha_E"] \arrow[r, "\pi"'] & T \end{tikzcd}$$

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For each JC^*-triple E there is a largest TRO $T^*(E)$ generated by (triple isomorphic copies of) E

$$\begin{tikzcd}
T^*(E) \arrow[dr, dashed, "\tilde{\pi}"] & \\
E \arrow[u, "\alpha_E"] \arrow[r, "\pi"'] & T
\end{tikzcd}$$

Definition

A JC^*-triple E is called *universally reversible* if $\pi(E)$ is reversible for each triple hom $\pi\colon E \to \mathcal{B}(\mathcal{K})$.

$\exists$ canonical ternary involution $\phi_E\colon T^*(E) \to T^*(E)$ fixing $\alpha_E(E)$.

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Theorem

If $E = T$ a TRO, then E is universally reversible $\iff$ $\nexists$ TRO homs from T onto row or column Hilbert spaces of dimension ≥ 3. If $\nexists$ on any dimension bar dimension 2, $T^*(E) = T \oplus T^{\mathrm{op}}$.

Example

$E = \mathbb{M}_{n,m}(\mathbb{C}) \subset T^*(E) = \mathbb{M}_{n,m}(\mathbb{C}) \oplus \mathbb{M}_{m,n}(\mathbb{C})$ via $x \mapsto x \oplus x^t$ if $\min(n, m) > 1$.

In this case, given any JC^*-triple $F \subseteq \mathcal{B}(K)$ and a linear isometry $\pi : E \twoheadrightarrow F$

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$\mathrm{TRO}(F) \cong T^*(E)/\ker\tilde{\pi}$ and only 3 valid $\ker\tilde{\pi}$: $\{0\}$, $\{0\} \oplus \mathbb{M}_{m,n}$, $\mathbb{M}_{n,m} \oplus \{0\}$.

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Note that if U is universally reversible, so are summands U_{l} and U_{r}. In fact U_{s} is always universally reversible.
We can also pass easily from involutions ϕ of a TRO T to bidual.

Theorem

If T is a TRO, T universally reversible as a JC^*-triple, ϕ a ternary involution of T, then T^ϕ is universally reversible unless there is ternary hom $\pi\colon T \twoheadrightarrow \mathbb{M}_n(\mathbb{C})$ with $n = 3$ or 4 and $\pi(\phi(x)) = -(\pi(x))^t$.

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Remark

(Conversely) If E is a universally reversible JC^*-triple, then $T = T^*(E)$ has a canonical involution ϕ with $E = T^\phi$ — and T must be universally reversible.

Proof depends on results charactierising universal reversibility of JC^*-triples in terms of 'factor' representations.

There are 4 classes of (Cartan) factors:

1. $E = \mathcal{B}(H, K)$ (or $E = \mathcal{B}(\mathcal{H})p$ up to isometry)
2. $E = \{x \in \mathcal{B}(\mathcal{H}) : x^t = x\}$ (dim $H > 1$) $S_{\dim H}$
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4. V_n spin factors, spanned by the identity and n 'spins' (= anticommuting (selfadjoint) unitaries with square the identity). ($n \geq 2$)

All Cartan factor JC^*-triples are dual spaces.

A factor representation is $\pi : E \to C$, triple hom (for $\{\cdot,\cdot,\cdot\}$) with weak*-dense range.

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Theorem (JLMS 2013)

If E is a JC^*-triple, E universally reversible $\iff$ it has no factor representations onto Hilbert spaces of dimension ≥ 3 or V_n for $n \geq 4$.

If U is a JW^*-triple (dual space, or has a weak-closed realisation in $\mathcal{B}(\mathcal{H})$), need only consider weak*-continuous representations onto factors.*

E is univerally reversible $\iff$ E^{**} is.

Since $\pi : U \to C$ weak-continuous has $\ker \pi$ a weak*-closed ideal, $U = (\ker \pi) \oplus_\infty (\ker \pi)^\perp$.*

Cartan factors contain minimal tripotents, ones where

$$E_2(e) = \{x \in E : 2\{e, e, x\} = ee^*x + xe^*e = 2x\}$$

has $\dim E_2(e) = 1$.

Corollary

Can rephrase using (factor) ideals in E^{**} generated by minimal tripotents.

For $E = T$ a TRO, V_n ruled out (restate).

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Cartan factors contain minimal tripotents, ones where

$$E_2(e) = \{x \in E : 2\{e, e, x\} = ee^*x + xe^*e = 2x\}$$

has $\dim E_2(e) = 1$.

Corollary

Can rephrase using (factor) ideals in E^{**} generated by minimal tripotents.
For $E = T$ a TRO, V_n ruled out (restate).

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Theorem

If T is a TRO, T universally reversible as a JC^*-triple, ϕ a ternary involution of T, then T^ϕ is universally reversible unless there is ternary hom $\pi\colon T \twoheadrightarrow \mathbb{M}_n(\mathbb{C})$ with $n = 3$ or 4 and $\pi(\phi(x)) = -(\pi(x))^t$.

Idea for proof.

Pass to bidual $U = T^{**}$. Extend ϕ. Easy to see $(T^\phi)^{**} = U^\phi$. Recall

$$U^\phi \cong (U_{\mathrm{s}})^\phi \oplus U_{\mathrm{r}}$$

Look at minimal tripotents $e \in (U_{\mathrm{s}})^\phi$. Either minimal in U_{s} or the sum of two minimals f, g in U_{s} exchanged by ϕ.
Weak*-closed ideals of U_{s} generated by f and g may be the same or exchanged by ϕ. Must be Type I. □

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Application

Theorem (Solel 2001)

Let $\pi\colon U \to V$ be a surjective linear isometry between W^*-TROs. Then there are $\pi_1, \pi_2\colon U \to V$ with π_1 a TRO homomorphism, π_2 a TRO anti-homomorphism, $\pi_1(U) \perp \pi_2(U)$ and $\pi = \pi_1 + \pi_2$. Moreover there is a central projection z in the left W^*-algebra $\overline{\mathscr{L}_V}$ of V with $\pi_1(x) = z\pi(x)$ for $x \in U$.

Proof in one case.

If U is univerally reversible with no 1-dim reps, we know $T^*(U) = U \oplus U^{\mathrm{op}}$.

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