# Fixed points of ternary involutions and applications

Richard M. Timoney Trinity College Dublin

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Joint work with Les Bunce

## Definition

A TRO is a norm closed linear subspace  $\mathcal{T}\subseteq\mathcal{B}(\mathcal{H})$  such that

$$x, y, z \in T \Rightarrow [x, y, z] := xy^*z \in T$$

#### Examples

$$T = A$$
.  $\mathbb{M}_n(T)$ .  $T = \mathbb{M}_{n,m}(\mathbb{C})$ .  $T = pAq$ .  $T^{\mathrm{op}} \subseteq \mathcal{B}(\mathcal{H})^{\mathrm{op}}$ .

#### Notation

 $\mathcal{L}_T = \operatorname{span}\{xy^* : x, y \in T\} \qquad \qquad \mathcal{R}_T = \operatorname{span}\{y^*z : y, z \in T\}$ Linking C\*-algebra of T:

$$\mathfrak{L}_{\mathcal{T}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathscr{L}_{\mathcal{T}} & \mathcal{T} \\ \mathcal{T}^* & \mathscr{R}_{\mathcal{T}} \end{pmatrix} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

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 $I \subseteq T$  is an *ideal* if it is a norm closed linear subspace with

 $[I, T, T] + [T, I, T] + [T, T, I] \subseteq I$ 

Since  $x \in I \Rightarrow x = [y, y, y]$  for some  $y \in I$ , can omit [T, I, T] (or require only  $[T, I, T] \subseteq I$ ).

#### Proposition

 $I \subseteq T$  an ideal implies  $\mathscr{R}_I \subseteq \mathscr{R}_T$  an ideal (and so is  $\mathscr{L}_I \subseteq \mathscr{L}_T$ ). Moreover

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 $\phi: T_1 \to T_2$  is a ternary homomorphism if  $\phi[x, y, z] = [\phi(x), \phi(y), \phi(z)]$  (or  $\phi(xy^*z) = \phi(x)(\phi(y))^*\phi(z)$ ).

#### Proposition

Ternary homomorphisms are (completely) contractive.  $\phi: T_1 \to T_2$  induces \*-homomorphisms  $\mathscr{L}_{\phi}: \mathscr{L}_{T_1} \to \mathscr{L}_{T_2}$  $(xy^* \mapsto \phi(x)(\phi(y))^*)$  and  $\mathscr{R}_{\phi}: \mathscr{R}_{T_1} \to \mathscr{R}_{T_2}$  and

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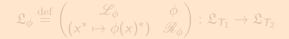
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 Abstract TRO:  $(T, [\cdot, \cdot, \cdot])$ .

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For 
$$p = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{L}_{T}^{\sim}, q = 1 - p = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$
  
 $\mathcal{T} \cong \begin{pmatrix} 0 & \mathcal{T} \\ 0 & 0 \end{pmatrix} = p(\mathfrak{L}_{T})q$ 

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 $e \in T$  is called a tripotent if  $[e, e, e] = ee^*e = e$  (  $\iff$  e a partial isometry)

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If we consider TROs  $U \in \mathcal{B}(\mathcal{H})$  that are weak\*-closed (or are Banach dual spaces), all extreme points of the unit ball are tripotents.

Bidual or weak\* closure of T can be U. Use  $\overline{\mathscr{L}_U} = \operatorname{span}\{xy^* : x, y \in U\}^{W^*}$ ,  $\overline{\mathscr{R}_U}$  and  $\begin{pmatrix}\overline{\mathscr{L}_U} & U\\ U^* & \overline{\mathscr{R}_U}\end{pmatrix} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ 

#### Proposition (Zettl)

Weak\*-closed 'ideals'  $I \subseteq U$  are in 1-1 correspondence with projections  $z \in Z(\overline{\mathscr{R}_U})$  via I = Uz.

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If we consider TROs  $U \in \mathcal{B}(\mathcal{H})$  that are weak\*-closed (or are Banach dual spaces), all extreme points of the unit ball are tripotents.

Bidual or weak\* closure of T can be U. Use  $\overline{\mathscr{L}_U} = \overline{\operatorname{span}\{xy^* : x, y \in U\}}^{w^*}$ ,  $\overline{\mathscr{R}_U}$  and  $\begin{pmatrix} \overline{\mathscr{L}_U} & U \\ U^* & \overline{\mathscr{R}_U} \end{pmatrix} \subseteq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ 

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A  $W^*$ -TRO U is called a *left TRO* if U is TRO isomorphic to Wp for  $p = p^* = p^2 \in W$ , W a  $W^*$ -algebra. U is called *square* if  $U \cong W$ . U square-free if  $\not\supseteq I \subseteq U$  with  $I \neq \{0\}$  square.

U a W\*-TRO implies

 $\textit{U} = \textit{U}_{l} \oplus \textit{U}_{r} \oplus \textit{U}_{s}$ 

with  $U_l/U_r/U_s$  the largest square-free left/ square-free right/ square weak\*-closed ideals of U.

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An *involution* of a C\*-algebra A is  $\Phi: A \to A$  such that  $\Phi$  is  $\mathbb{C}$ -linear,  $\Phi(\Phi(a)) = a$ ,  $\Phi(ab) = \Phi(b)\Phi(a)$ , and  $\Phi(a^*) = \Phi(a)^*$ 

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A ternary involution of a TRO T is  $\phi: T \to T$  C-linear,  $\phi(\phi(a)) = a, \ \phi[a, b, c] = [\phi(c), \phi(b), \phi(a)]$  (or  $\phi(ab^*c) = \phi(c)(\phi(b))^*\phi(a)).$ 

Fixed points  $T^{\phi} = \{x \in T : \phi(x) = x\} = \{x + \phi(x) : x \in T\}.$  $A^{\Phi}$  will be a (closed) Jordan \*-algebra of operators (*JC*\*-algebra).  $T^{\phi}$  will be a *JC*\*-triple: closed under Jordan triple product

$$\{a, b, c\} \stackrel{\text{def}}{=} ([a, b, c] + [c, b, a])/2 = (ab^*c + cb^*a)/2$$

In fact  $T^{\phi}$  is reversible:

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A  $JC^*$ -triple is a closed linear  $E \subseteq \mathcal{B}(\mathcal{H})$  such that

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#### Examples

$$E = T$$
 or  $E = T^{\phi}$  (e.g. with  $T = \mathbb{M}_n(\mathbb{C}), \ \phi(x) = x^t$  or  $\phi(x) = -x^t$ ).

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Relate to isometric theory of Banach spaces (since triple homomorphisms  $\pi: E \to F \subseteq \mathcal{B}(\mathcal{K})$  are contractive).

 $(E, \{\cdot, \cdot, \cdot\})$  abstract triple has no canonical op. space structure.

Neal & Russo found that for many E, there are only a few.

#### Example

TROs *T* give rise to at least 3 obvious concrete *JC*\*-triples: E = T,  $E = T^{op}$  and  $E = \{x \oplus x^{op} : x \in T\} \subseteq T \oplus T^{op}$ 

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These examples are reversible. In latter case  $E = (T \oplus T^{op})^{\phi}$ where  $\phi(x \oplus y^{op}) = y \oplus x^{op}$ .

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A *JC*<sup>\*</sup>-triple *E* is called *universally reversible* if  $\pi(E)$  is reversible for each triple hom  $\pi: E \to \mathcal{B}(\mathcal{K})$ .

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If E = T a TRO, then E is universally reversible  $\iff \nexists$  TRO homs from T onto row or column Hilbert spaces of dimension  $\ge 3$ . If  $\nexists$  on any dimension bar dimension 2,  $T^*(E) = T \oplus T^{\text{op}}$ .

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 $E = \mathbb{M}_{n,m}(\mathbb{C}) \subset T^*(E) = \mathbb{M}_{n,m}(\mathbb{C}) \oplus \mathbb{M}_{m,n}(\mathbb{C}) \text{ via } x \mapsto x \oplus x^t \text{ if } \min(n,m) > 1.$ In this case, given any  $JC^*$ -triple  $F \subseteq \mathcal{B}(K)$  and a linear isometry  $\pi \colon E \twoheadrightarrow F$ 



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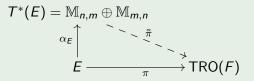


TRO(F)  $\cong T^*(E)/\ker \tilde{\pi}$  and only 3 valid ker  $\tilde{\pi}$  : {0}, {0}  $\oplus \mathbb{M}_{m,n}$ ,  $\mathbb{M}_{n,m} \oplus \{0\}$ .

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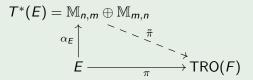
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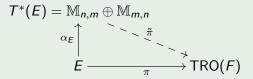
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Proof depends on results charactierising universal reversibility of  $JC^*$ -triples in terms of 'factor' representations. There are 4 classes of (Cartan) factors:

- $E = \mathcal{B}(H, K)$  (or  $E = \mathcal{B}(\mathcal{H})p$  up to isometry)
- $E = \{ x \in \mathcal{B}(\mathcal{H}) : x^t = x \} (\dim H > 1)$ S<sub>dim H</sub>

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All Cartan factor *JC*\*-triples are dual spaces.

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If E is a JC\*-triple, E universally reversible  $\iff$  it has no factor representations onto Hilbert spaces of dimension  $\ge 3$  or  $V_n$  for  $n \ge 4$ .

If U is a JW\*-triple (dual space, or has a weak\*-closed realisation in  $\mathcal{B}(\mathcal{H})$ ), need only consider weak\*-continuous representations onto factors.

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Since  $\pi: U \to C$  weak\*-continuous has ker  $\pi$  a weak\*-closed ideal,  $U = (\ker \pi) \oplus_{\infty} (\ker \pi)^{\perp}$ .

Cartan factors contain minimal tripotents, ones where

$$E_2(e) = \{x \in E : 2\{e, e, x\} = ee^*x + xe^*e = 2x\}$$

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Look at minimal tripotents  $e \in (U_s)^{\phi}$ . Either minimal in  $U_s$  or the sum of two minimals f, g in  $U_s$  exchanged by  $\phi$ .

Weak\*-closed ideals of  $U_{\rm s}$  generated by f and g may be the same or exchanged by  $\phi$ . Must be Type I.

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