Simple amenable operator algebras

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Wrong Definition of Strong Self-Absorption

I claimed unital $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing if:

- $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$
- The flip map is approximately inner on $\mathcal{D} \otimes \mathcal{D}$.

But in fact

Unital $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing if:

- The flip map is approximately inner on $\mathcal{D} \otimes \mathcal{D}$.

and either $\boxed{\mathcal{D} \cong \mathcal{D}^{\otimes\infty}}$ or a certain central sequence condition holds

The point is that this gives an isomorphism $\theta : \mathcal{D} \xrightarrow{\cong} \mathcal{D} \otimes \mathcal{D}$ which is approx unitarily equivalent to $x \mapsto x \otimes 1_{\mathcal{D}}$

However: just assuming $\mathcal{D}$ has approximate inner flip

Proof that $\mathcal{D} \hookrightarrow A_\omega \cap A' \Rightarrow A \cong A \otimes \mathcal{D}$ still true

Converse holds when $\mathcal{D}$ is strongly self-absorbing.

Jiang's theorem: unital $\mathcal{Z}$-stable C^*-algebras are K_1 injective

$[u]_1 = 0 \quad u \in \mathcal{U}(A) \Rightarrow u \sim 1$ in $\mathcal{U}(A)$.

For A unital and M_{n^∞}-stable

- A is K_1-injective
- $K_0(A)$ generated by $\{[p]_0 : p \in \mathcal{P}(A)\}$.

Recall $\mathcal{Z}$ is an inductive limit of $\mathcal{Z}_{2^\infty,3^\infty}$'s

$= \{ f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty}) : f(0) \in M_{2^\infty} \otimes 1,\ f(1) \in 1 \otimes M_{3^\infty} \}$

- It suffices to show $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-injective
- Fix unitary $u \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ with $[u]_1 = 0$.

$$0 \to A \otimes SM_{6^\infty} \to \underset{\substack{\cup\!\!\!\!\shortmid \\ u}}{A \otimes \mathcal{Z}_{2^\infty,3^\infty}} \xrightarrow{q} A \otimes (M_{2^\infty} \oplus M_{3^\infty}) \to 0,$$

$= A \otimes M_{2^\infty} \oplus A \otimes M_{3^\infty}$

$[q(u)]_1 = 0 \quad \therefore \quad q(u) \sim_h 1$ in $A \otimes (M_{2^\infty} \oplus M_{3^\infty})$

lift $q(u)$ to $v \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ replace u by $v^* u$ so $q(u) = 1$.

Jiang's theorem

- Fix unitary $u \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ with $[u]_1 = 0$.
- wlog $q(u) = 1$, so $u \in (A \otimes SM_{6^\infty})^\sim$ $= \{ f \in C(\mathbb{T}, A \otimes M_{6^\infty}) : f(1) \in \mathbb{C} \}$

$$\begin{array}{ccccc}
K_1(A \otimes SM_{6^\infty}) & \longrightarrow & K_1(A \otimes \mathcal{Z}_{2^\infty,3^\infty}) & \longrightarrow & K_1(A \otimes (M_{2^\infty} \oplus M_{3^\infty})) \\
{\scriptstyle \exp}\uparrow \quad [u]_1 & \mapsto & [u]_1 = 0 & & \downarrow \\
K_0(A \otimes (M_{2^\infty} \oplus M_{3^\infty})) & \longleftarrow & K_0(A \otimes \mathcal{Z}_{2^\infty,3^\infty}) & \longleftarrow & K_0(A \otimes SM_{6^\infty})
\end{array}$$

Claim

Can replace u so that $[u]_1 = 0$ in $K_1(A \otimes SM_{6^\infty})$.

$\therefore [u]_1 = \exp(x)$, $x \in K_0(A \otimes (M_{2^\infty} \oplus M_{3^\infty})) = \sum_{j=1}^{n} ([p_i]_0 - [q_i]_0)$

with $p_i, q_i \in \mathcal{P}(A \otimes (M_{2^\infty} \oplus M_{3^\infty}))$.

lift these to self adjoints $h_i, k_i \in (A \otimes \mathcal{Z}_{2^\infty,3^\infty})$ in $K_1(A \otimes SM_{6^\infty})$

$$w = u e^{-2\pi i h_1} \cdots e^{-2\pi i h_n} e^{2\pi i k_1} \cdots e^{2\pi i k_n} \sim u \qquad \& [w]_1 = 0$$

Jiang's theorem

- Fix unitary $u \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ with $[u]_1 = 0$.

$$0 \to A \otimes SM_{6^\infty} \to A \otimes \mathcal{Z}_{2^\infty,3^\infty} \xrightarrow{q} A \otimes (M_{2^\infty} \oplus M_{3^\infty}) \to 0,$$

- wlog $q(u) = 1$, so $u \in (A \otimes SM_{6^\infty})$
- and wlog $[u]_1 = 0$ in $K_1(A \otimes SM_{6^\infty})$

$\therefore$ $[u]_1 = 0$ in $C(\mathbb{T}, A \otimes M_{6^\infty})$ which is K_1-injective

$\therefore$ $\exists$ path (v_t) $v_0 = 1$, $v_1 = u$ in $C(\mathbb{T}, A \otimes M_{6^\infty})$

$u_t = v_t(1)^* v_t$ path from 1 to u in $(A \otimes SM_{6^\infty})^\sim$

Recall: Matui-Sato. Lift McDuffness to tracially large order zero map

$A \neq M_n$, simple nuclear with unique trace. $\pi_\tau(A)'' \cong \mathcal{R}$

$$\begin{tikzcd} A_\omega \cap A' \arrow[r, two heads] & \mathcal{R}^\omega \cap \mathcal{R}' \\ & M_n \arrow[u] \arrow[ul, "\varphi \text{ o/z}"] \end{tikzcd}$$

$\tau_\omega(\varphi(1)) = 1$

- What if A has more than one trace?

For each $\tau \in T(A)$ $\exists\, \varphi_\tau : M_n \to A_\omega \cap A'$ o/z $\quad \tau_\omega(\varphi_\tau(1)) = 1$.

Want tracially large o/z map $\varphi : M_n \to A_\omega \cap A'$ o/z. $\quad \tau(\varphi(1)) = 1$ $\forall \tau \in T_\omega(A)$.

Need to look at all traces simulateously, and obtain uniform estimates.

Looking at all the traces at the same time

- $\pi_\tau(A)''$ doesn't carry uniform information about all traces on A.
- A_{fin}^{**} sees all traces — but not uniformly.

Recall

Let τ be a trace on a C^*-algebra A. Then $\pi_\tau(A)$ is a von Neumann algebra iff the unit ball of A is complete in $\|\cdot\|_{2,\tau}$.

Definition $A = C(X)$ $\|x\|_{2,T(A)} = \|x\|$

Let A be a C^*-algebra with $T(A) \neq \emptyset$. $\|x\|_{2,T(A)} = \sup_{\tau \in T(A)} \|x\|_{2,\tau}$

$$\overline{A}^{T(A)} := \frac{\{\text{norm bounded, } \|\cdot\|_{2,T(A)}\text{-Cauchy sequences}\}}{\{\text{norm bounded, } \|\cdot\|_{2,T(A)}\text{-null sequences}\}}$$ Ozawa.

- Tracial completion of A. $\|\cdot\|_{2,T(A)}$ extends to $\overline{A}^{T(A)}$

$$\overline{A}^{T(A)} := \frac{\{\text{norm bounded, } \|\cdot\|_{2,T(A)}\text{-Cauchy sequences}\}}{\{\text{norm bounded, } \|\cdot\|_{2,T(A)}\text{-null sequences}\}}$$

- Tracial completion of A. $\|\cdot\|_{2,T(A)}$ extends to $\overline{A}^{T(A)}$.

DEFINITION (CCEGSTW) $\mathcal{M}$ C^*-alg,

A tracially complete C^*-algebra is a pair $(\mathcal{M}, X)$ such that $X \subset T(\mathcal{M})$ is a closed convex set such that

- $\|x\|_{2,X} = \sup_{\tau \in X} \tau(x^*x)^{1/2}$ is a norm on $\mathcal{M}$.
- The unit ball of $\mathcal{M}$ is complete in $\|\cdot\|_{2,X}$.

eg $(\underset{\text{VNA}}{\mathcal{M}}, \underset{\text{faithful}}{\{\tau\}})$; $(\underset{\text{finite VNA}}{\mathcal{M}}, T(\mathcal{M}))$, $(\overline{A}^{T(A)}, T(A))$.

finite VNAs $\subseteq$ tracially complete C^* $\subseteq$ C^*-algs.

- $\theta: (\mathcal{M}, X) \to (\mathcal{N}, Y)$ s.t. for $\tau \in Y$, $\tau \circ \theta \in X$

McDuffness (again)

Various operations: follow constructions for finite vNa using $\|\cdot\|_{2,X}$ rather than $\|\cdot\|_{2,\tau}$.

- $(\mathcal{M},X)\,\overline{\otimes}\,(\mathcal{N},Y) = \overline{(\mathcal{M}\odot\mathcal{N})}^{\overline{co}(X\times Y)}$.
- $(\mathcal{M},X)^\omega$ has algebra $\mathcal{M}^\omega = \ell^\infty(\mathcal{M})/\{(x_n) : \lim_{n\to\omega}\|x_n\|_{2,X} = 0\}$.

THEN

for $\|\cdot\|_{2,X}$-separable tracially complete C^*-algebras:

$$(\mathcal{M},X) \cong (\mathcal{M},X)\,\overline{\otimes}\,(\mathcal{R},\{\tau_{\mathcal{R}}\}) \iff M_n \hookrightarrow (\mathcal{M},X)^\omega \cap \mathcal{M}'.$$

Call these McDuff tracially complete C^*-algs.

Lifting argument goes through as well [$(\bar{A}^{T(A)}, T(A))$ is McDuff then $\exists$ $\varphi: M_n \to A_\omega \cap A'$ tracially large.

- Is also A simple nuclear & has almost unperforated Cuntz semigroup : $\bar{A}^{T(A)}$ McDuff $\Rightarrow A \cong A\otimes\mathcal{Z}$.

↑ Matui-Sato.

Open: $\bar{A}^{T(A)}$ is McDuff? (In this generality) eg AF A s.t. $T(A) = X$ where $\partial_e T(X) = \{\tau_1, \tau_2, \tau_3, \dots\}$

$$\tau_n \to \frac{1}{2}(\tau_1 + \tau_2)$$

$$\bar{A}^{T(A)} = \{ (x_n) \in \ell^\infty(\mathbb{R}) \mid x_n \to \theta \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \}.$$

Fix $\theta : M_2(\mathbb{R}) \xrightarrow{\cong} \mathbb{R}$.

From pointwise to uniform?

From

$$\forall \tau \in T_\omega(A),\ \exists \phi_\tau : M_n \to (A_\omega \cap A'),\ \text{such that } \tau(\varphi(1)) = 1$$

To

$$\exists \phi : M_n \to (A_\omega \cap A'),\ \text{such that } \forall \tau \in T_\omega(A),\ \tau(\varphi(1)) = 1.$$

Another eg: For a unitary $u \in (\mathcal{M}, X)$

For each $\tau \in X$, $\varepsilon > 0$, $\exists\, h_\tau$ self-adjoint s.t. $\|u - e^{ih_\tau}\|_{2,\tau} < \varepsilon$, $\|h_\tau\| \leq \pi$

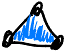
Borel calculus in $\pi_\tau(\mathcal{M})''$.

Qn: $\forall \varepsilon > 0\ \exists\, h = h^*$ s.t. $\|u - e^{ih}\|_{2,X} < \varepsilon$?

(i.e. unitaries in $\mathcal{M}^\omega$ are exponentials & $K_1(\mathcal{M}^\omega) = 0$).

Find $\exists\, h_1, \ldots, h_n$ s.t. $\forall \tau\ \exists j$ s.t. $\|u - e^{ih_j}\|_{2,\tau} < \varepsilon$.

Pointwise to uniform: McDuffness is universal (at least with a factor condition)

$T(\mathcal{U}) =$ 

Definition

$(\mathcal{M}, X)$ is factorial if X is a closed face of $T(\mathcal{M})$.

Automatic (but needs work) for $(\overline{A}^{T(A)}, T(A))$.

$T(A) \subseteq T(\overline{A}^{T(A)})$

Example — Theorem

factorial

Let $(\mathcal{M}, X)$ be a McDuff tracially complete C^*-algebra and $u \in \mathcal{M}$ unitary. Then there exists self-adjoint $h \in \mathcal{M}^\omega$ with $u = e^{ih}$.

- eg $(A^{\overline{T(A)}}, T(A))$ with A $\mathcal{Z}$-stable.

Pointwise to uniform: McDuffness is universal

A classification type example

A consequence of Connes' theorem

Let A be a separable nuclear C^*-algebra and $\mathcal{M}$ a finite von Neumann algebra. Maps $A \to \mathcal{M}$ are classified by traces.

- $\varphi, \psi : A \to \mathcal{M}$ $\tau \circ \varphi = \tau \circ \psi$ $\forall \tau \in T(\mathcal{M}) \Rightarrow \varphi \approx_{au} \psi$. (point strong $*$)
- Given cts affine $\alpha : T(\mathcal{M}) \to T(A)$ $\exists\, \varphi : A \to \mathcal{M}$ s.t. $\tau \circ \varphi = \alpha(\tau)$ for $\tau \in T(\mathcal{M})$

Uniform trace version

Let A be separable nuclear C^*-algebra, and $(\mathcal{M}, X)$ a McDuff factorial tracially complete C^*-algebra. Maps $A \to \mathcal{M}$ are classified by traces.

- $\varphi, \psi : A \to \mathcal{M}$ s.t. $\tau \circ \varphi = \tau \circ \psi$ $\forall \tau \in X \Rightarrow$ $\varphi \approx_{au} \psi$.
- Given $X \xrightarrow[\text{affine cts}]{} T(A)$ $\exists\, \varphi : A \to \mathcal{M}$ s.t. $\tau \circ$ $= \alpha(\tau)$.

AMENABILITY FOR TRACIALLY COMPLETE C^*-ALGEBRAS

$\mathcal{M}, X \xrightarrow{\theta} \mathcal{N}, Y$ $\quad$ $\tau \circ \theta \in X$ whenever $\tau \in Y$

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "\theta"] \arrow[dr, "\phi_i"', "\text{cpc}"] & & \mathcal{N} \\
& F_i \arrow[ur, "\psi_i"', "\text{cpc}"] &
\end{tikzcd}$$

$\|\psi_i \phi_i(x) - x\|_{2,X} \to 0$ for all $x \in \mathcal{M}$.

THEOREM (CCEGSTW)

- Amenable McDuff factorial tracially complete C^*-algebras are approximately finite dimensional.
- They are then classified by the specified set of traces.

A, B $\mathcal{Z}$-stable, nuclear, $T(A), T(B) \neq \emptyset$ sep, unital.

$\bar{A}^{T(A)} \cong \bar{B}^{T(B)} \iff T(A) \cong T(B)$.

A amenable rep

Tracially complete $\longrightarrow$ McDuff tracially complete