SIMPLE AMENABLE OPERATOR ALGEBRAS

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McDuff's characterisation

Theorem (McDuff '69)

Let $\mathcal{M}$ be a separably acting II_1 factor. TFAE:

1. $\mathcal{M}$ is McDuff, i.e. $\mathcal{M} \cong \mathcal{M} \overline{\otimes} \mathcal{R}$.
2. $\mathcal{M}^\omega \cap \mathcal{M}'$ is non-abelian
3. $\mathcal{R} \hookrightarrow \mathcal{M}^\omega \cap \mathcal{M}'$
4. $M_n \hookrightarrow \mathcal{M}^\omega \cap \mathcal{M}'$

Proof (3$\Rightarrow$ 1) is via an abstract intertwining argument.
Two ingredients:

- $\mathcal{R} \cong \mathcal{R} \overline{\otimes} \mathcal{R}$
- The tensor flip $x \otimes y \mapsto y \otimes x$ is approximately inner.

Now for C^*-algebras: strong self absorption

$A_\omega = \ell^\infty(A) / \{(x_n) \mid \lim_{n\to\omega} \|x_n\| = 0\}$.

- Works mutatis mutandis to characterise absorption of UHF algebras of infinite type eg $A \cong A \otimes M_{2^\infty} \Leftrightarrow M_{2^\infty} \hookrightarrow A_\omega \cap A'$

Definition

Unital $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing if:

- $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$
- The flip map is approximately inner on $\mathcal{D} \otimes \mathcal{D}$.

$A \cong A \otimes \mathcal{D} \Leftrightarrow \mathcal{D} \hookrightarrow A_\omega \cap A'$

& $\exists \theta : A \xrightarrow{\cong} A \otimes \mathcal{D}$ s.t. $\theta(a) \approx_{au} a \otimes 1_{\mathcal{D}}$

egs UHF of infinite type, $\mathcal{O}_2$, $\mathcal{O}_\infty$, $\mathcal{O}_\infty \otimes$ UHF of infinite type, $\mathcal{Z}$.

CONSEQUENCES OF APPROX INNER FLIP ON $\mathcal{D} \otimes \mathcal{D}$

- $\mathcal{D}$ is simple — exercise
- $\mathcal{D}$ has at most one trace $\tau_1, \tau_2 \in T(D)$

$$\tau_1(x) = (\tau_1 \otimes \tau_2)(x \otimes 1) = (\tau_1 \otimes \tau_2)(u_n (x \otimes 1) u_n^*)$$

$$\downarrow$$

$$(\tau_1 \otimes \tau_2)(1 \otimes x) = \tau_2(x).$$

- $\mathcal{D}$ is nuclear.

In $(B \otimes_{min} D) \otimes_{max} D$ $\quad x = \sum x_i \otimes y_i \in B \odot D$

$$\| x \otimes 1 \| = \| (1 \otimes u_n)(x \otimes 1)(1 \otimes u_n^*) \| \quad u_n \in \mathcal{U}(D \otimes D)$$

$$\downarrow$$

$$\| \sum x_i \otimes 1 \otimes y_i \| \quad \square.$$

THE JIANG-SU ALGEBRA $\mathcal{Z}$

WHAT IS THE RIGHT ANALOG OF $\mathcal{R}$?

- M_{2^∞}? Too uncanonical.
- $\mathcal{Q} = \bigotimes_{n=2}^{\infty} M_n$? Too big. $M_{2^\infty} \otimes \mathcal{Q} \cong \mathcal{Q}$.

PROPERTIES OF $\mathcal{Z}$

- Strongly self-absorbing with $K_0(\mathcal{Z}) = \mathbb{Z}$ (and $K_1(\mathcal{Z}) = 0$). eg $\mathcal{O}_\infty$.
- $\mathcal{Z}$ has a trace (which is then necessarily unique).

$A \cong A \otimes \mathcal{Z}$ necessary for classification by K-theory and traces.

- Trace on $A \otimes \mathcal{Z}$ must be of the form $\tau_A \otimes \tau_{\mathcal{Z}}$ for $\tau_A \in T(A)$
- $K_*(A \otimes \mathcal{Z}) \cong K_*(A)$,

McDuff's characterisation for $\mathcal{Z}$

A is $\mathcal{Z}$-stable $\iff \mathcal{Z} \hookrightarrow A_\omega \cap A'$ $\quad$ $M_2 \hookrightarrow A_\omega \cap A'$?

Order zero maps

A c.p.c. map $\phi : A \to B$ is order zero if it preserves orthogonality, i.e. $xy = 0$ (say in A_{sa}) implies $\phi(x)\phi(y) = 0$.

Liftability i.e. is F is finite dim &

$$\begin{tikzcd} & B \arrow[d, two heads] \\ F \arrow[ur, dotted, "\text{o/z lift}"] \arrow[r, "\text{o/z}"'] & B/J \end{tikzcd}$$

$\boxed{\varphi \text{ unital} \Rightarrow \varphi \text{ is a } *\text{-hom.}}$

Definition

Order zero $\phi : M_n \to B$ is large if exists $v \in B$ with $1_B - \phi(1_{M_n}) = v^*v$ and $\phi(e_{11})vv^* = vv^*$.

Thm: A sep. TFAE
1/ $A \cong A \otimes \mathcal{Z}$
2/ $\exists\, n \geq 2$ & large order zero map $\varphi : M_n \to A_\omega \cap A'$
3/ $\forall\, n \geq 2$ & large order zero map $\varphi : M_n \to A_\omega \cap A'$

$\mathcal{Z}$-STABILITY AND COMPARISON.

CUNTZ COMPARISON OF POSITIVE ELEMENTS (IN $\bigcup_n M_n(A)$)

- $a \precsim b \iff \exists (x_m),\ x_m^* b x_m \to a.$
- $a \sim b \iff a \precsim b$ and $b \precsim a.$

$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

Cuntz semigroup of A is almost unperforated if $a^{\oplus(n+1)} \precsim b^{\oplus n} \Rightarrow a \precsim b.$

THEOREM (RØRDAM)

A ~~simple~~, unital $\mathcal{Z}$-stable, implies the Cuntz semigroup of A is almost unperforated.

Given $\varphi : M_n \to M_k(A)_\omega \cap M_k(A)'$ $\qquad a^{\oplus(n+1)} \lesssim b^{\oplus n}$ in $M_k(A)$

Wlog o/z map.

$\varphi(e_{11})^{\oplus n} \sim \varphi(1) \lesssim 1 = 1 - \varphi(1) + \varphi(1)$

$\lesssim \varphi(e_{11})^{\oplus(n+1)}$

$\varphi(e_{ii}) \sim \varphi(e_{jj})$
i,j

$a \lesssim (a\,\varphi(e_{11}))^{\oplus(n+1)} \sim \varphi(e_{11})^{1/2} a^{\oplus(n+1)} \varphi(e_{11})^{1/2}$

$\lesssim \varphi(e_{11})^{1/2} b^{\oplus n} \varphi(e_{11})^{1/2}$

$$a \lesssim (a\,\varphi(e_{11}))^{\oplus(n+1)} \sim \varphi(e_{11})^{1/2}\, a^{\oplus(n+1)}\, \varphi(e_{11})^{1/2}$$

$$\lesssim \varphi(e_{11})^{1/2}\, b^{\oplus n}\, \varphi(e_{11})^{1/2}$$

$$\varphi(e_{11})^{\oplus n} \lesssim 1 \qquad \sim (b\,\varphi(e_{11}))^{\oplus n} \lesssim b.$$

Measuring largeness in trace

A sequence $(\tau_n)_{n=1}^{\infty}$ of traces on A induces a limit trace τ on A_ω:

$$\tau((x_n)) = \lim_{n\to\omega} \tau_n(x_n). \qquad (x_n) \in \ell^\infty(A)$$

Write $T_\omega(A)$ for the set of all limit traces.

$\varphi: M_n \to A_\omega \cap A'$ tracially large $\iff$ $\tau(\varphi(1)) = 1$ $\forall \tau \in T_\omega(A)$

Thm (Matui-Sato '12): A simple unital nuclear with almost unperforated Cuntz semigroup. Then $\varphi: M_n \to A_\omega \cap A'$ tracially large $\Rightarrow$ φ is large.

- Suppose also that A has ! trace.

• Suppose also that A has ! trace, then $\pi_\tau(A)'' = \mathcal{R}$.

Then (deep fact)

$$\begin{tikzcd} A_\omega \cap A' \arrow[r, two heads] & \mathcal{R}^\omega \cap \mathcal{R}' \\ & M_n \arrow[u, "\theta"', "\text{unital } *\text{-hom}"] \arrow[ul, "\varphi"] \end{tikzcd}$$

θ unital $*$-hom, as $\mathcal{R}$ is McDuff.

φ o/zero lifting.

As φ lifts θ, $\tau_\omega(\varphi(1)) = 1$ so φ is tracially large & A is $\mathcal{Z}$-stable.

A PICTURE OF $\mathcal{Z}$

$Z_{p^\infty,q^\infty} = \{ f \in C([0,1], M_{p^\infty} \otimes M_{q^\infty}) : f(0) \in M_{p^\infty} \otimes 1, \ f(1) \in 1 \otimes M_{q^\infty} \}$

A MODERN 'CONSTRUCTION'

$\mathcal{Z}$ is the unique inductive limit

$$Z_{p^\infty,q^\infty} \xrightarrow{\theta} Z_{p^\infty,q^\infty} \xrightarrow{\theta} Z_{p^\infty,q^\infty} \xrightarrow{\theta} \ldots$$

with θ standard.

p, q coprime.

for any $\tau \in T(Z_{p^\infty,q^\infty})$

$$(\tau \circ \theta)(f) = \int_0^1 \mathrm{tr}_{M_{p^\infty} \otimes M_{q^\infty}}(f(t))\,dt$$

Schemaitat '19.

JIANG'S THEOREM

THEOREM

A unital $\mathcal{Z}$-stable C^*-algebra A is K_1-injective.

$u \in \mathcal{U}(A) \quad [u]_1 = 0 \quad \Rightarrow \quad u \sim_h 1 \quad \text{in } \mathcal{U}(A).$

- First consider case when A is M_{n^∞}-stable.

$u \oplus 1^{\oplus n^k - 1} \sim_h 1^{\oplus n^k} \qquad u^{\oplus n^k} \sim 1^{\oplus n^k}$

$$\begin{array}{ccccc} u \otimes 1_{n^k} & \sim_h & 1_A \otimes 1_{n^k} & \text{in} & \mathcal{U}(A \otimes M_{n^k}) \\ \approx \downarrow & & \downarrow & & \downarrow \cong \\ u & \sim_h & 1 & & A \end{array}$$

EXERCISE

For M_{n^∞}-stable A, $K_0(A)$ is generated by $\{[p]_0 : p \text{ a projection in } A\}$.

Jiang's theorem

Recall $\mathcal{Z}$ is an inductive limit of $\mathcal{Z}_{2^\infty,3^\infty}$'s

- It suffices to show $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is K_1-injective
- Fix unitary $u \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ with $[u]_1 = 0$.

$$0 \to A \otimes SM_{6^\infty} \to A \otimes \mathcal{Z}_{2^\infty,3^\infty} \xrightarrow{q} A \otimes (M_{2^\infty} \oplus M_{3^\infty}) \to 0,$$

Jiang's theorem

- Fix unitary $u \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ with $[u]_1 = 0$.
- wlog $q(u) = 1$, so $u \in (A \otimes SM_{6^\infty})$

$$
\begin{tikzcd}
K_1(A \otimes SM_{6^\infty}) \arrow[r] & K_1(A \otimes \mathcal{Z}_{2^\infty,3^\infty}) \arrow[r] & K_1(A \otimes (M_{2^\infty} \oplus M_{3^\infty})) \arrow[d] \\
K_0(A \otimes (M_{2^\infty} \oplus M_{3^\infty})) \arrow[u, "\exp"] & K_0(A \otimes \mathcal{Z}_{2^\infty,3^\infty}) \arrow[l] & K_0(A \otimes SM_{6^\infty}) \arrow[l]
\end{tikzcd}
$$

Claim

Can replace u so that $[u]_1 = 0$ in $K_1(A \otimes SM_{6^\infty})$.

Jiang's theorem

- Fix unitary $u \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ with $[u]_1 = 0$.

$$0 \to A \otimes SM_{6^\infty} \to A \otimes \mathcal{Z}_{2^\infty,3^\infty} \xrightarrow{q} A \otimes (M_{2^\infty} \oplus M_{3^\infty}) \to 0,$$

- wlog $q(u) = 1$, so $u \in (A \otimes SM_{6^\infty})$
- and wlog $[u]_1 = 0$ in $K_1(A \otimes SM_{6^\infty})$