# SIMPLE AMENABLE OPERATOR ALGEBRAS

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## Unital classification theorem (many hands)

Simple, separable, unital, nuclear, $\mathcal{Z}$-stable $C^*$-algebras satisfying the UCT are classified by $K$-theory and traces.

- Analogue for $C^*$-algebras of the Murray-von Neumann, Connes, Haagerup classification of injective von Neumann factors.
- 25+ year endeavour; work of many researchers.

## Goals

- Look at some aspects of structure and classification of $C^*$-algebras through lens of comparison with von Neumann algebras.
  - What von Neumann algebras should we use?
- Particular focus on tensorial absorption '$\mathcal{Z}$-stability'

*These days it is common for young operator algebraists to know a lot about $C^*$-algebras, or a lot about von Neumann algebras – but not both. Though a natural consequence of the breadth and depth of each subject, this is unfortunate as the interplay between the two theories has deep historical roots and has led to many beautiful results. We review some of these connections, in the context of amenability, with the hope of convincing (younger) readers that tribalism impedes progress.*

Nate Brown
The symbiosis of $C^*$- and $W^*$-algebras, arXiv:0812.1763

# Factors

- A factor is a von Neumann algebra with a trivial centre.
- Factors are simple von Neumann algebras.

$I \triangleleft \mathcal{U}$ weak*-closed 2-sided ideal. Then $I = p\,\mathcal{U}$
a projection $p \in Z(\mathcal{U})$. $\mathcal{U} = \oplus\int_\Omega$ (factors) ; $L^\infty[0,1] = \oplus\int_0^1 \mathbb{C}\, d\mu$.

| Type I | Type $\mathrm{II}_1$ | Type $\mathrm{II}_\infty$ | Type III |
|---|---|---|---|
| • $I_n$ $M_n(\mathbb{C})$ <br> • $I_\infty$ $B(H)$ | not $M_n$ <br> • has a trace $\tau$ <br> $\tau(xy) = \tau(yx)$. | $\mathrm{II}_1 \otimes B(H)$. | All non-zero projections are equivalent |

Projections classified by trace

$p \sim q \Leftrightarrow tr(p) = tr(q)$

# SIMPLE $C^*$-ALGEBRAS

- No non-trivial closed two sided ideals.
- Quasi-central approximate unit for $I \lhd A$

| Elementary | Stably Finite | Purely Infinite $A$ |
|---|---|---|
| $M_n$<br>$K$ | All projections in $A \otimes K$ are finite | For all $a, b \geq 0$ $a, b \neq 0$<br>$\forall \varepsilon > 0$ $\exists x \in A$ s.t.<br>$\| x^* a x - b \| < \varepsilon$. |

↳ Rørdam '01 $\exists$ simple nuclear $C^*$-algebra with both finite & infinite projections – this is not a tensor product as in the $II_\infty$-case.

# Amenability

- $C^*$-algebra $A$ is nuclear
- von Neumann algebra $A$ is semidiscrete if $\exists$ cpc maps

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\varphi_i"'] & \approx & A \\ & F_i \arrow[ur, "\psi_i"'] & \end{tikzcd}$$

$\leftarrow$ finite dimensional

- $\|\psi_i(\varphi_i(x)) - x\| \to 0$ in $\| \ \|$ when $A$ is $C^*$.
- $\psi_i(\varphi_i(x)) - x \to 0$ weak$^*$ when $A$ is VNA

for $x \in A$

## Theorem

$A$ nuclear $\iff$ $A^{**}$ is semidiscrete

$\Leftarrow$ Hahn Banach argument.

$\Rightarrow$ Difficult Connes.

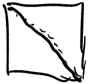
# McDuff factors

$$\mathcal{R} = \left(\bigotimes_1^\infty M_2\right)'' \cong \left(\bigotimes_1^\infty M_2\right)'' \bar{\otimes} \left(\bigotimes_1^\infty M_2\right)'' = \mathcal{R} \bar{\otimes} \mathcal{R}.$$

### Definition

A separably acting $\mathrm{II}_1$ factor $\mathcal{M}$ is McDuff, if $\mathcal{M} \cong \mathcal{M} \overline{\otimes} \mathcal{R}$.

Given $\mathcal{F} \subset \mathcal{M}$, $\varepsilon > 0$ $\exists\, \theta : \underset{\mathcal{M} \bar{\otimes} \mathcal{R}}{\overset{\shortparallel}{\mathcal{M}}} \to \underset{\mathcal{M} \bar{\otimes} \mathcal{R} \bar{\otimes} \mathcal{R}}{\overset{\shortparallel}{\mathcal{M} \bar{\otimes} \mathcal{R}}}$ s.t. $\theta(x) \approx_\varepsilon x \otimes 1_{\mathcal{R}}$.

$\theta(x) =$ 

i.e. $\|\theta(x) - x \otimes 1_{\mathcal{R}}\|_{2,\tau} < \varepsilon$

$\|y\|_{2,\tau}^2 = \tau(y^* y).$

### Example of what can be proved from $\theta(x) \approx x \otimes 1$

McDuff factors are singly generated.

# McDuff's criterion ('69)

$\mathcal{M}$ is McDuff iff for every finite subset $\mathcal{F} \subset \mathcal{M}$ and $\epsilon > 0$, $\exists$ unital $\phi : M_2 \to \mathcal{M}$ such that $\|[\phi(e_{i,j}), x]\|_2 < \epsilon$ for $x \in \mathcal{F}$.

Fix $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ $\quad \mathcal{M}^\omega = \{(x_n) \in \ell^\infty(\mathcal{M})\} / \{(x_n) : \lim_{n\to\omega} \|x_n\|_2 = 0\}$

This has a faithful trace $\tau_\omega((x_n)) = \lim_{n\to\omega} \tau(x_n)$ & $\mathcal{M}^\omega$ is a VNA.

$\mathcal{M} \hookrightarrow \mathcal{M}^\omega$ as const sequence. $\rightsquigarrow \mathcal{M}^\omega \cap \mathcal{M}'$

$\pi_\tau(A) = \pi_\tau(A)''$

**Lemma**

Let $\tau$ be a trace on a $C^*$-algebra $A$. Then $\pi_\tau(A)$ is a von Neumann algebra iff the unit ball of $A$ is complete in $\|\cdot\|_{2,\tau}$.

| Non $\Gamma$ | $\Gamma$ not McDuff | McDuff |
|---|---|---|
| $\mathcal{M}^\omega \cap \mathcal{M}' = \mathbb{C}1$ | $\mathcal{M}^\omega \cap \mathcal{M}' \neq \mathbb{C}1$ & abelian. In this case it has no minimal projs. | $\mathcal{M}^\omega \cap \mathcal{M}'$ not abelian $\Leftrightarrow \mathcal{M}^\omega \cap \mathcal{M}'$ is $\mathrm{II}_1$ (not necessary a factor) $\Leftrightarrow R \hookrightarrow \mathcal{M}^\omega \cap \mathcal{M}'$ $\Leftrightarrow M_n \hookrightarrow \mathcal{M}^\omega \cap \mathcal{M}'$ $\forall n \geq 2$. |

# PROVING $\mathcal{R} \hookrightarrow \mathcal{M}^\omega \cap \mathcal{M}' \implies \mathcal{M}$ MCDUFF

## AN ABSTRACT INTERTWINING ARGUMENT

Let $A, B$ be separable, $\phi : A \hookrightarrow B$. Suppose $\exists$ unitaries $(v_n)_n$ in $B$ st

- $[v_n, \phi(a)] \to 0$ for $a \in A$.
- $\mathrm{dist}(v_n^* b v_n, \phi(A)) \to 0$ for $b \in B$.

Then $\phi$ is approximately unitarily equivalent to an isomorphism.

$$\begin{tikzcd}
\mathcal{F}_1 \Subset A \arrow[r] \arrow[d, "v_{k_1}\phi(\,)v_{k_1}^*"'] \arrow[dr, phantom, "\not\approx"] & A \supset \mathcal{F}_2 \arrow[r] \arrow[d, "v_{k_2}v_{k_1}\phi(\,)v_{k_1}^*v_{k_2}^* \cdots"] & A \arrow[r] & A \arrow[r] & A \arrow[r] & \cdots \arrow[r] & A \arrow[d, "\lim_{i\to\infty} v_{k_i}\cdots v_{k_1}\phi(\,)v_{k_1}^*\cdots v_{k_i}^*"] \\
\mathcal{G}_1 \Subset B \arrow[r] & B \arrow[r] & B \arrow[r] & B \arrow[r] & \cdots & & B
\end{tikzcd}$$

$\mathcal{G}_2 \Subset B$ (second column)

# Proving $\mathcal{R} \hookrightarrow \mathcal{M}^\omega \cap \mathcal{M}' \implies \mathcal{M}$ McDuff

- Let $\phi : \mathcal{M} \to \mathcal{M} \overline{\otimes} \mathcal{R}$ be $\phi(x) = x \otimes 1_{\mathcal{R}}$.
- Fix $\theta : \mathcal{R} \to \mathcal{M}^\omega \cap \mathcal{M}'$.

$$\theta \otimes id : \quad R \bar{\otimes} R \longrightarrow (M \otimes R)^\omega \cap (M \otimes 1_R)'$$

$$x \otimes y \mapsto \theta(x)(1 \otimes y)$$

Flip map on $R \otimes R$ approx inner, i.e. $\exists\, u_n \in \mathcal{U}(R \otimes R)$

$x \otimes y \mapsto y \otimes x$. $\qquad u_n(x \otimes y)u_n^* \to y \otimes x$

$$v_n = (\theta \otimes id)(u_n) \in (M \otimes R)^\omega \cap (M \otimes 1_R)'$$

$$v_n^*(m \otimes y)v = v_n^*(m \otimes 1)(1 \otimes y)v_n \approx (m \otimes 1)v_n^*(1 \otimes y)v_n$$

$$\approx m\,\theta(y) \otimes 1 \in \phi(M).$$

$\square$.