BMO spaces in von Neumann algebras

Joint work with Martijn Caspers

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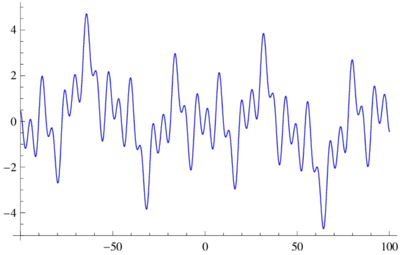
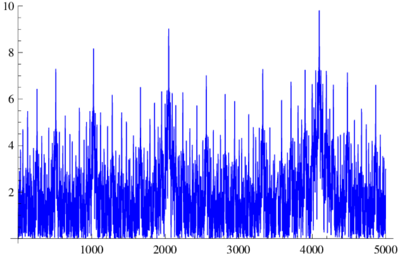
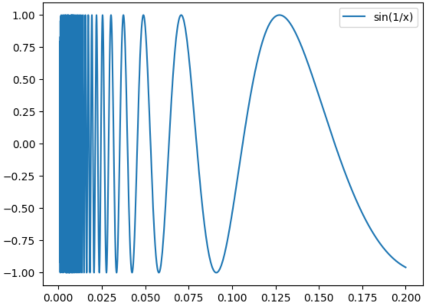
Overview

1. Classical BMO spaces
2. Noncommutative L_p spaces
3. Noncommutative BMO spaces
4. A predual for BMO
5. Interpolation

Classical BMO spaces







Classical BMO spaces

"Mean difference to the average". If Q is a cube in $\mathbb{R}^n$, set

$$f_Q = \frac{1}{|Q|}\int_Q f dx.$$

$$\text{"mean oscillation over } Q\text{"} = \frac{1}{|Q|}\int_Q |f - f_Q|^2 dx$$

$$\|f\|_{\mathrm{BMO}} = \sup_Q \left(\frac{1}{|Q|}\int_Q |f - f_Q|^2 dx\right)^{1/2}$$

$$\mathrm{BMO} = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathrm{BMO}} < \infty\}.$$

$\|f\|_{\mathrm{BMO}} = 0$ iff f is constant

- $H_1^* \cong \mathrm{BMO}$. - Fefferman/Stein duality
- $[L_1, \mathrm{BMO}]_{1/p} \cong L_p$ - Complex interpolation

Preliminaries

Definition

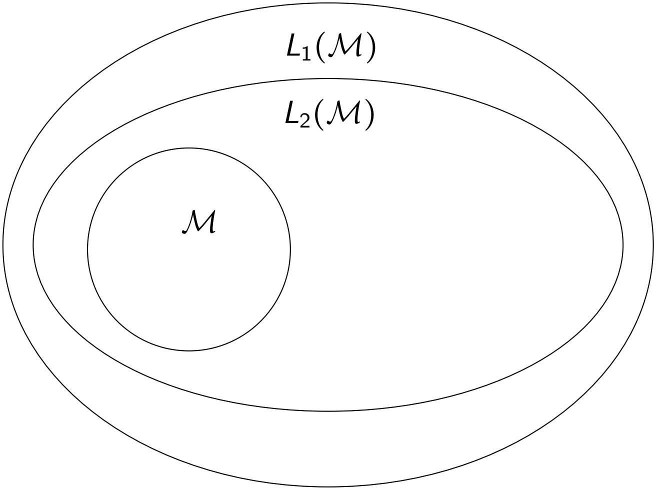
A von Neumann algebra $\mathcal{M}$ is called σ-**finite** if there exists a normal faithful (n.f.) state $\varphi \in \mathcal{M}^*$. It is called **finite** if there exists a n.f. *tracial* state $\tau \in \mathcal{M}^*$.

Example of a σ-finite von Neumann algebra:

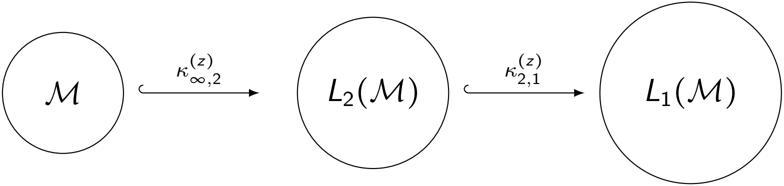
- Quantum group $SU_q(2)$, von Neumann algebra $L_\infty(SU_q(2))$, n.f. Haar state φ

L_p spaces in finite von Neumann algebras

$$L_p(\mathcal{M},\tau) = \{x \in \overline{\mathcal{M}} \mid \|x\|_p := \tau(|x|^p)^{1/p} < \infty\}.$$



L_p spaces in σ-finite von Neumann algebras



- $z \in [-1, 1]$
- $\kappa_{q,p}^{(z)}(x)^* = \kappa_{q,p}^{(-z)}(x^*)$
- $(\kappa_{2,1}^{(1)} \circ \kappa_{\infty,2}^{(-1)})(x) = (\kappa_{2,1}^{(-1)} \circ \kappa_{\infty,2}^{(1)})(x) = \kappa_{\infty,1}^{(0)}(x)$

History of noncommutative BMO spaces

- (Pisier-Xu, '97) - Martingale BMO spaces, $(H_1)^* - \mathrm{BMO}$ duality. Requires filtration of von Neumann algebra.
- (Mei, '08), (Junge-Mei, '12) - Semigroup BMO spaces for finite von Neumann algebras, interpolation results, 'abstract' $(h_1)^* - \mathrm{BMO}$ duality.
- (Caspers, '19) - Semigroup BMO for σ-finite von Neumann algebras, interpolation results. Uses a 'smaller' BMO space.

Markov semigroups

Definition

Let $(\mathcal{M}, \varphi)$ be a von Neumann algebra. A (GNS-symmetric) **Markov semigroup** is a semigroup of linear operators $(T_t)_{t\geq 0} : \mathcal{M} \to \mathcal{M}$ satisfying

i) T_t is normal ucp, $t \geq 0$,
ii) $\varphi(T_t(x)y) = \varphi(xT_t(y))$, $x, y \in \mathcal{M}$, $t \geq 0$ (GNS-symmetry)
iii) The mapping $t \mapsto T_t(x)$ is strongly continuous, $x \in \mathcal{M}$.

It is called φ-**modular** if $T_t \circ \sigma_s^\varphi = \sigma_s^\varphi \circ T_t$ for all $s, t \in \mathbb{R}$.

The T_t can be extended to $L_p(\mathcal{M})$.

BMO for finite von Neumann algebras

Classical case: $\|f\|_{\mathrm{BMO}} = \sup_Q \left(\frac{1}{|Q|}\int_Q |f - f_Q|^2 dx\right)^{1/2}$

$(\mathcal{M}, \tau)$ finite von Neumann algebra.

Definition

Let $(T_t)_{t\geq 0}$ be a Markov semigroup on $\mathcal{M}$. The **column** respectively **row BMO norm** is defined as

$$\|x\|_{\mathrm{BMO}^c} = \sup_{t\geq 0} \|T_t(|x - T_t(x)|^2)\|_\infty^{\frac{1}{2}}, \quad \|x\|_{\mathrm{BMO}^r} = \|x^*\|_{\mathrm{BMO}^c}.$$

The **BMO norm** is given by

$$\|x\|_{\mathrm{BMO}} = \max\{\|x\|_{\mathrm{BMO}^c}, \|x\|_{\mathrm{BMO}^r}\}.$$

Also defined for $x \in L_p(\mathcal{M})$, $2 \leq p < \infty$.

These are seminorms!

BMO for finite von Neumann algebras

Need to remove (among others) multiples of the identity.

$$L_p^\circ(\mathcal{M}) := \{x \in L_p(\mathcal{M}) : T_t(x) \to 0\}, \quad 1 \leq p < \infty,$$

$$\mathcal{M}^\circ := \{x \in \mathcal{M} : T_t(x) \to 0 \text{ in the weak-}* \text{ topology.}\}$$

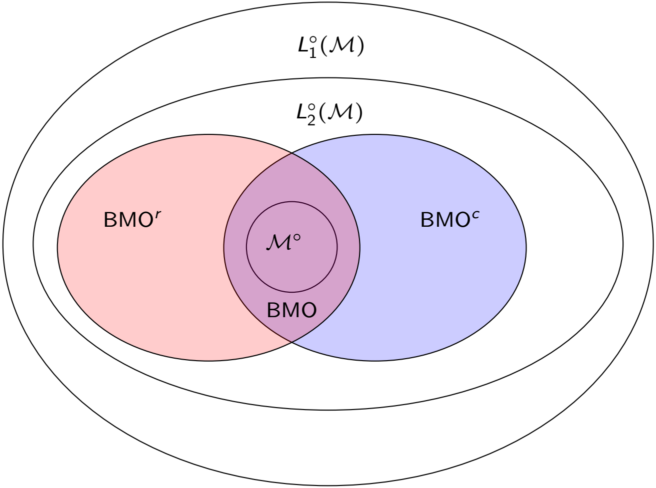
BMO is a norm on $L_p^\circ(\mathcal{M})$, $2 \leq p < \infty$. The BMO spaces are defined as

$$\text{BMO} = \{x \in L_2^\circ(\mathcal{M}) : \|x\|_{\text{BMO}} < \infty\},$$

and similarly for BMO^c, BMO^r

BMO for finite von Neumann algebras

$$\|x\|_{\mathrm{BMO}} = \max\{\|x\|_{\mathrm{BMO}^c}, \|x\|_{\mathrm{BMO}^r}\}.$$

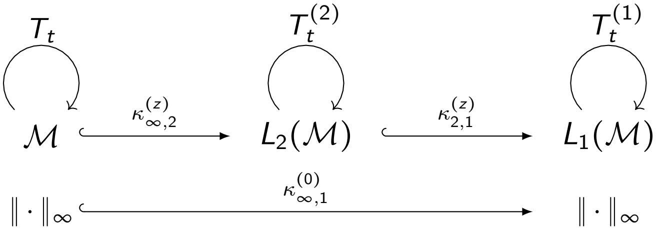


BMO for σ-finite von Neumann algebras

Finite case:

$$\|x\|_{\mathrm{BMO}^c} = \sup_{t\geq 0} \| T_t(|x - T_t(x)|^2) \|_\infty^{\frac{1}{2}}, \quad x \in L_2(\mathcal{M})$$

From now on, $(\mathcal{M}, \varphi)$ is a σ-finite von Neumann algebra.



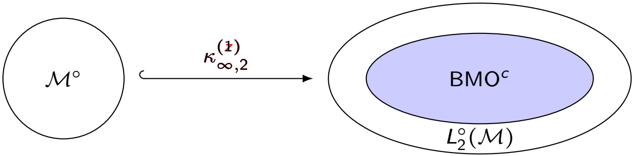
We henceforth assume $(T_t)_{t\geq 0}$ to be φ-modular!

$$\|x\|_{\mathrm{BMO}^c} = \sup_{t\geq 0} \| T_t^{(1)}(|x - T_t^{(2)}(x)|^2) \|_\infty^{\frac{1}{2}}, \quad x \in L_2(\mathcal{M})$$

BMO for σ-finite von Neumann algebras

Again, $\|\cdot\|_{\mathrm{BMO}^c}$ is a norm on $L_2^\circ(\mathcal{M})$.

$$\mathrm{BMO}^c = \{x \in L_2^\circ(\mathcal{M}) : \|x\|_{\mathrm{BMO}^c} < \infty\}.$$



We can also define $\|\cdot\|_{\mathrm{BMO}^c}$ on $\mathcal{M}$!

$$\|x\|_{\mathrm{BMO}^c} = \sup_{t\geq 0} \|T_t(|x - T_t(x)|^2)\|_\infty^{\frac{1}{2}}, \quad x \in \mathcal{M}$$

BMO for σ-finite von Neumann algebras

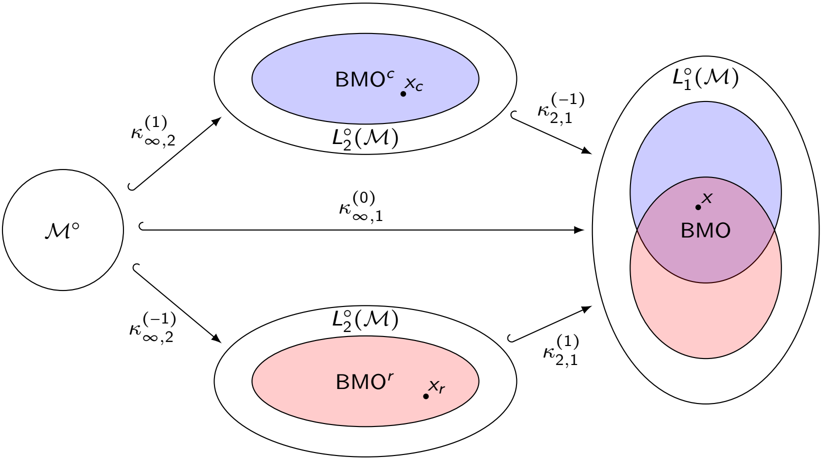
$$\|x\|_{\mathrm{BMO}^c} = \|\kappa_{\infty,2}^{(1)}(x)\|_{\mathrm{BMO}^c}, \qquad \|x\|_{\mathrm{BMO}^r} = \|\kappa_{\infty,2}^{(-1)}(x)\|_{\mathrm{BMO}^r}$$

How to define $\|\cdot\|_{\mathrm{BMO}}$? For $x \in \mathcal{M}$, we have

$$\|x\|_{\mathrm{BMO}} = \max\{\|x\|_{\mathrm{BMO}^c}, \|x\|_{\mathrm{BMO}^r}\}$$

but if we do this for $x \in L_2^\circ(\mathcal{M})$, then there is no embedding preserving the BMO-norm!

BMO for σ-finite von Neumann algebras



$$\|x\|_{\mathrm{BMO}} = \max\{\|x_c\|_{\mathrm{BMO}^c}, \|x_r\|_{\mathrm{BMO}^r}\}.$$

A predual for BMO^c

Goal: create 'abstract' predual h_r^1 for BMO^c. For $y \in L_2^\circ(\mathcal{M})$:

$$\|y\|_{h_1^r} = \sup_{\|x\|_{\mathrm{BMO}^c} \leq 1} |\langle y, x^* \rangle|.$$

We set

$$h_1^r = \overline{L_2^\circ(\mathcal{M})}^{\|\cdot\|_{h_1^r}}.$$

A predual for BMO

Proposition

$$BMO^c \cong (h_1^r)^*, \quad BMO^r \cong (h_1^c)^*$$

Theorem (See Bergh-Löfström)

Let (A_0, A_1) *be a compatible couple such that* $A_0 \cap A_1$ *is dense in* A_0 *and* A_1*. Then*

$$(A_0 + A_1)^* \cong A_0^* \cap A_1^*.$$

Idea: find *suitable* compatible couple structure for (h_1^r, h_1^c), set $h_1 = h_1^r + h_1^c$.

A predual for BMO

Theorem (Caspers-V.)

$(\mathcal{M}, \varphi)$ *σ-finite. Set* $h_1 = h_1^c + h_1^r$. *Then* $(h_1)^* \cong BMO$.

Corollary

BMO is a Banach space.

Interpolation

Theorem (Junge-Mei '12)

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and assume $(T_t)_{t \geq 0}$ is a Markov semigroup on $\mathcal{M}$ that admits a standard Markov dilation. Then

$$L^{\circ}_{pq}(\mathcal{M}) = [BMO, L^{\circ}_{p}(\mathcal{M})]_{1/q}$$

Theorem (Caspers '19, Caspers-V.)

Let $(\mathcal{M}, \varphi)$ be a σ-finite von Neumann algebra and assume $(T_t)_{t \geq 0}$ is a φ-modular Markov semigroup on $\mathcal{M}$ that admits a standard Markov dilation. Then

$$L^{\circ}_{pq}(\mathcal{M}) = [BMO, L^{\circ}_{p}(\mathcal{M})]_{1/q}$$