Maximal C^*-covers and residual finite-dimensionality

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Outline

1. C^*-covers and their partial ordering

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2. Equate this partial ordering as one arising from the spectrum of a C^*-algebra
3. Applications to residual finite-dimensionality of operator algebras

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and

$$\omega : \mathrm{A}(\mathbb{D}) \hookrightarrow \mathfrak{T}, \quad f \mapsto M_f.$$

Here, $\mathfrak{T}$ is the Toeplitz algebra and $\mathfrak{K}(H^2) \subset \mathfrak{T}$.

Partial ordering

Fix an operator algebra $\mathcal{A}$. Define a relation $\preceq$ on the C^*-covers of $\mathcal{A}$ by $(\mathfrak{A}, \iota) \preceq (\mathfrak{B}, j)$ if and only if there is a surjective $*$-homomorphism

$$\pi : \mathfrak{B} \to \mathfrak{A}, \qquad \pi \circ j = \iota.$$

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Minimal and maximal C^*-covers

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The maximal C^*-cover can be shown to exist by taking

$$\mu = \bigoplus_{\rho \text{ c.c. repn}} \rho$$

Universal property

Given a completely contractive representation $\rho : \mathcal{A} \to B(\mathcal{K})$, there is a unique $*$-representation $\theta : \mathrm{C}^*_{max}(\mathcal{A}) \to B(\mathcal{K})$ such that

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Hence, there is a 1:1 correspondence between c.c. representations of $\mathcal{A}$ and $*$-representations of $\mathrm{C}^*_{max}(\mathcal{A})$.

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Example (Blecher)

If $\mathcal{A}$ is the 2×2 upper triangular matrices and $\mathcal{A}_0 \subset \mathcal{A}$ are the matrices with constant diagonal, then $\mathrm{C}^*_{max}(\mathcal{A}_0)$ is the universal C^*-algebra generated by a nilpotent operator.

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If $\mathcal{A}$ is the 2×2 upper triangular matrices and $\mathcal{A}_0 \subset \mathcal{A}$ are the matrices with constant diagonal, then $\mathrm{C}^*_{max}(\mathcal{A}_0)$ is the universal C^*-algebra generated by a nilpotent operator. In addition,

$$\mathrm{C}^*_{max}(\mathcal{A}) \cong \{f \in \mathrm{C}([0,1], \mathbb{M}_2) : f(0) \text{ diagonal matrix}\}.$$

Spectrum of a C^*-algebra

For a C^*-algebra $\mathfrak{A}$, let

$$\widehat{\mathfrak{A}} = \{\text{unitary equivalences of irreducible representations of } \mathfrak{A}\}$$

denote the spectrum of $\mathfrak{A}$. A topology on $\widehat{\mathfrak{A}}$ is given by declaring the open subsets to be of the form

$$\mathcal{U}_{\mathfrak{J}} = \{[\sigma] \in \widehat{\mathfrak{A}} : \sigma \mid_{\mathfrak{J}} \neq 0\}$$

where $\mathfrak{J}$ is a closed two-sided ideal of $\mathfrak{A}$.

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where $\mathfrak{J}$ is a closed two-sided ideal of $\mathfrak{A}$. In fact,

$$\mathcal{U}_{\mathfrak{J}} \simeq \widehat{\mathfrak{J}}, \qquad \widehat{\mathfrak{A}} \setminus \mathcal{U}_{\mathfrak{J}} \simeq \widehat{\mathfrak{A}/\mathfrak{J}}.$$

Spectra of C^*-covers

Notation: If $(\mathfrak{A}, \iota)$ is a C^*-cover of $\mathcal{A}$, then there is a surjective $*$-representation

$$q_{\mathfrak{A}} : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathfrak{A}, \qquad q_{\mathfrak{A}} \circ \mu = \iota.$$

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Let $(\mathfrak{A}, \iota)$ be a C^*-cover and define

$$\mathcal{S}(\mathfrak{A}, \iota) := \{[\sigma] \in \widehat{\mathrm{C}^*_{max}(\mathcal{A})} : \sigma \mid_{\ker q_{\mathfrak{A}}} = 0\} \simeq \widehat{\mathfrak{A}}.$$

So $\widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ inherits the spectra of all C^*-covers as closed subspaces.

Topology and Order

Theorem (T. 2021)

Let $X \subset \widehat{\mathrm{C}^*_{max}(\mathcal{A})}$ be some subset. Then,

$$X = \mathcal{S}(\mathfrak{A}, \iota) \Leftrightarrow X \text{ is closed and contains } \mathcal{S}(\mathrm{C}^*_e(\mathcal{A}), \iota_{env}).$$

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Proposition (T. 2021)

$(\mathfrak{A}, \iota) \preceq (\mathfrak{B}, j) \Leftrightarrow \mathcal{S}(\mathfrak{A}, \iota) \subset \mathcal{S}(\mathfrak{B}, j)$

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$\preceq$ *induces a complete lattice as follows: If* $\mathcal{C} = \{(\mathfrak{A}_\lambda, \iota_\lambda)\}_\lambda$ *are* C^**-covers, then*

- $\sup \mathcal{C} = (\mathrm{C}^*(\iota(\mathcal{A})), \iota)$ *where* $\iota = \bigoplus \iota_\lambda$
- $\inf \mathcal{C} = (\mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}, q \circ \mu)$ *where* $\mathfrak{J} = \overline{\sum \ker q_{\mathfrak{A}_\lambda}}$ *and* $q : \mathrm{C}^*_{max}(\mathcal{A}) \to \mathrm{C}^*_{max}(\mathcal{A})/\mathfrak{J}$ *is the quotient map.*

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Theorem (T. 2021)

There is a complete lattice isomorphism between the C^*-covers of $\mathcal{A}$ and $\{\mathcal{S}(\mathfrak{A}, \iota) : (\mathfrak{A}, \iota)\ \mathrm{C}^*\text{-cover of } \mathcal{A}\}$ given by $(\mathfrak{A}, \iota) \mapsto \mathcal{S}(\mathfrak{A}, \iota)$.

Residual finite-dimensionality

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Clouâtre-Dor-On (2021): some algebras from semigroups and analytic function spaces

RFD-maximal C^*-cover

Theorem (T. 2021)

If $\mathcal{A}$ is RFD, then there is a largest RFD C^*-cover of $\mathcal{A}$ (w.r.t $\preceq$), denoted $(\mathfrak{R}(\mathcal{A}), \upsilon)$. Further,

$$\mathcal{S}(\mathfrak{R}(\mathcal{A}), \upsilon) = \overline{\{\textit{finite-dimensional irreps}\}} \subset \widehat{\mathrm{C}^*_{max}(\mathcal{A})}.$$

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Hence, $\mathrm{C}^*_{max}(\mathcal{A})$ is RFD if and only if $(\mathfrak{R}(\mathcal{A}), \upsilon)$ is the maximal C^*-cover.

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Hence, $\mathrm{C}^*_{max}(\mathcal{A})$ is RFD if and only if $(\mathfrak{R}(\mathcal{A}), \upsilon)$ is the maximal C^*-cover. We provide supporting evidence to the previous question by proving multiple instances in which $(\mathfrak{R}(\mathcal{A}), \upsilon)$ satisfies properties that $\mathrm{C}^*_{max}(\mathcal{A})$ satisfies.

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In particular, utilizing $(\mathfrak{R}(\mathcal{A}), \upsilon)$, we provide a generalization to Hadwin's characterization of separable RFD C^*-algebras to a non self-adjoint setting.

Hadwin's characterization of RFD C^*-algebras

Let $\{e_n\}_{n\geq 1}$ be an ONB for ℓ^2, P_n be the proj. onto span$\{e_1, \ldots, e_n\}$ and $\mathcal{M}_n = P_n B(\ell^2) P_n$.

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We say a c.c. representation $\rho : \mathcal{A} \to B(\ell^2)$ is $*$-**liftable** if there is a c.c. representation $\tau : \mathcal{A} \to \mathfrak{B}$ such that $\pi \circ \tau = \rho$.

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Theorem (Hadwin 2014)

Let $\mathfrak{A}$ be a separable C^*-algebra. Then, $\mathfrak{A}$ is RFD if and only if every unital $*$-representation $\sigma : \widetilde{\mathfrak{A}} \to B(\ell^2)$ is $*$-liftable.

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Theorem (T. 2021)

Let $\mathcal{A}$ be a separable operator algebra. Then, $\mathrm{C}^*_{max}(\mathcal{A})$ is RFD if and only if every c.c. representation $\rho : \mathcal{A} \to B(\ell^2)$ is $*$-liftable.

Thank You!