Orthogonality and Gateaux derivative of C^*-norm

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Theorem (Grover P.; Singla S., 2021)

Let $a \in \mathcal{A}$. Let $\mathcal{B}$ be a subspace of $\mathcal{A}$. Then a is Birkhoff-James orthogonal to $\mathcal{B}$ if and only if there exists $\phi \in \mathcal{S}_{\mathcal{A}}$ such that $\phi(a^*a) = \|a\|^2$ and $\phi(a^*b) = 0$ for all $b \in \mathcal{B}$.

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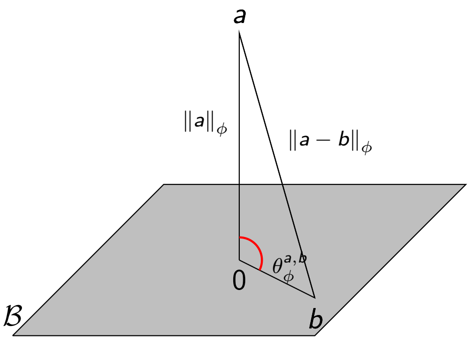

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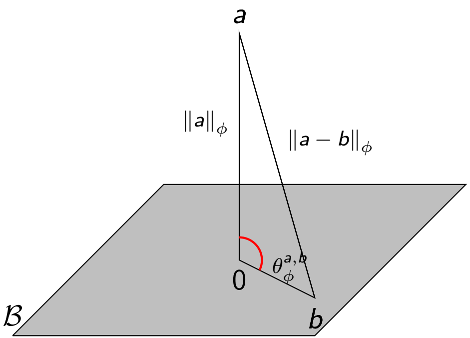

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Characterization of best approximation in $\mathcal{C}(X)$

Notation: $\mathbb{F}$ will stand for $\mathbb{C}$ or $\mathbb{R}$. Let $(\mathcal{C}(X), \|\cdot\|_\infty)$ be the space of $\mathbb{F}$-valued continuous functions on a compact Hausdorff space X.

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1. *g is a best approximation to f in W.*
2. *There exists a regular Borel probability measure μ on X such that*
   a) *the support of μ is contained in the set $\{x \in X : |(f-g)(x)| = \|f-g\|_\infty\}$ and*
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Let $\mathbb{M}_n(\mathbb{F})$ be the space of $n \times n$ matrices with entries in $\mathbb{F}$. A *density matrix* $A \in \mathbb{M}_n(\mathbb{F})$ is a non-negative matrix with $\mathrm{trace}\,(A) = 1$.

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Application to distance formulas

Theorem (Williams J. P., 1970)

For $a \in \mathcal{A}$, we have

$$\mathrm{dist}(a, \mathbb{C}1_{\mathcal{A}})^2 = \max\{\phi(a^*a) - |\phi(a)|^2 : \phi \in \mathcal{S}_{\mathcal{A}}\}.$$

Proof. There exists $\lambda_0 \in \mathbb{C}$ such that $\mathrm{dist}(a, \mathbb{C}1_{\mathcal{A}}) = \|a - \lambda_0 1_{\mathcal{A}}\|$.

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A generalization of this will be :

Theorem (Grover P.; Singla S., 2021)

Let $a \in \mathcal{A}$. Let $\mathcal{B}$ be a subspace of $\mathcal{A}$. Let b_0 be a best approximation to a in $\mathcal{B}$. Then

$$\operatorname{dist}(a, \mathcal{B})^2 = \max\{\phi(a^* a) - \phi(b_0^* b_0) : \phi \in S_{\mathcal{A}} \text{ and } \phi(a^* b) = \phi(b_0^* b) \text{ for all } b \in \mathcal{B}\}.$$

A distance formulas in terms of representation

Theorem (Grover P.; Singla S., 2021)

Let $a \in \mathcal{A}$. Let $\mathcal{B}$ be a subspace of $\mathcal{A}$. Suppose there is a best approximation to a in $\mathcal{B}$. Then

$$\mathrm{dist}(a, \mathcal{B}) = \max \Big\{ \big| \langle \pi(a)\xi | \eta \rangle \big| : (\mathcal{H}, \pi, \xi) \text{ is cyclic representation of } \mathcal{A}, \\ \eta \in \mathcal{H}, \|\eta\| = 1 \text{ and } \langle \pi(b)\xi | \eta \rangle = 0 \text{ for all } b \in \mathcal{B} \Big\}.$$

Proof. Clearly $RHS \leq LHS$.

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A distance formulas in terms of conditional expectation

Theorem (P. Grover, 2014)

Let $\mathcal{B}$ be C^*-subalgebra of $(\mathbb{C})$ containing idenity matrix. Let $\mathcal{C}_{\mathcal{B}}$ be orthogonal projection of $\mathbb{M}_n(\mathbb{C})$ onto $\mathcal{B}$.

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Let X be a Banach space, $x, y \in X$, and $\phi \in [0, 2\pi)$.

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Let $a, b \in \mathcal{A}$. *Then*

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Expression for Gateuax derivative

For a normed space V and $v, u \in V$, we have

$$\lim_{t \to 0^+} \frac{\|v + tu\| - \|v\|}{t} = \max\{\mathrm{Re}\, f(u) : f \in V^*, f(v) = \|v\|, \|f\| = 1\}.$$

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Corollary. For $A, B \in \mathscr{K}(\mathcal{H})$, we have

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Smooth points

We say that a vector v of norm one is a smooth point of the unit ball of V if there exists a unique functional F_v, called the support functional, such that $\|F_v\| = 1$ and $F_v(v) = 1$.

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Using $\lim\limits_{t\to 0^+} \dfrac{\|A+tB\|-\|A\|}{t} = \dfrac{1}{\|A\|} \max\limits_{\|u\|=1, A^*Au=\|A\|^2 u} Re\langle Au|Bu\rangle$, we get

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References

5. Bhatia R.; Šemrl P. : Orthogonality of matrices and some distance problems. *Linear Algebra Appl.* 287 (1999), 77–85.
6. Holub J. R. : On the metric geometry of ideals of operators on Hilbert space. *Math. Ann.* 201 (1973), 157–163.
7. Rieffel M. A. : Leibniz seminorms and best approximation from C^*-subalgebras. *Sci. China Math.* 54 (2011), 2259–2274.
8. Singer I. : Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces. *Springer-Verlag, Berlin,* 1970.
9. Williams J. P. : Finite operators. *Proc. Amer. Math. Soc.* 26 (1970), 129-136.

References

1. James R. C. : Orthogonality and linear functionals in normed linear spaces. *Trans. Amer. Math. Soc.* 61 (1947), 265–292.
2. Grover P. : Orthogonality to matrix subspaces, and a distance formula. *Linear Algebra Appl.* 445 (2014), 280–288.
3. Grover P. ; Singla S. : Best Approximations, distance formulas and orthogonality in C^*-algebras. *J. Ramanujan Math. Soc.* 36 (2021), 85–91.
4. Grover P. ; Singla S. : Birkhoff-James orthogonality and applications : A survey. *Operator Theory, Functional Analysis and Applications*, Birkhäuser, Springer, vol. 282, 2021.
5. Singla S. : Gateaux derivative of C^* norm. *communicated*.