Unitary groups & traces

- A, B will be unital, separable, & we'll assume they have non-empty trace simplices.
- $\mathcal{U}(A)$ = unitary group, $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$, $\mathcal{U}_\infty(A) = \varinjlim \mathcal{U}_n(A)$
- $T(A)$ is the trace simplex, $\mathrm{Aff}\,T(A)$ the cts affine functions on $T(A)$,

  $\rho_A : K_0(A) \longrightarrow \mathrm{Aff}\,T(A)$ the dual of the pairing.

Some results:

	$\mathcal{U}(A) \simeq \mathcal{U}(B)$	$A \simeq B$
(Sakai)	norm-cts iso	AW^*-factors
(Hatori, Molnár)	isometric metric spaces	Jordan * iso.
(Al-Rawashdeh, Booth, Giordano)	top. iso.	*-iso unital, AH-classifiable.

Let $\theta : \mathcal{U}^\circ(A) \longrightarrow \mathcal{U}^\circ(B)$ cts hom.

Using the de la Harpe-Skandalis determinant

$$\Delta_A : \mathcal{U}^\circ_\infty(A) \longrightarrow \frac{\mathrm{Aff}\,T(A)}{\rho_A(K_0(A))}.$$

$$u \longmapsto \tilde{\Delta}_A(\xi) + \rho_A(K_0(A))$$

$\xi : 1 \rightsquigarrow u$ piecewise smooth

$$\tilde{\Delta}_A(\xi)(\tau) = \int_0^1 \frac{1}{2\pi i}\tau(\xi'(t)\,\xi(t)^{-1})\,dt.$$

Theorem of Ng + Robert: If A is "nice": $\cdot \ker \Delta_A \cap \mathcal{U}^\circ(A) = D\mathcal{U}^\circ(A)$

$\cdot\ \pi_1(\mathcal{U}^\circ(A)) \longrightarrow \pi_1(\mathcal{U}^\circ_\infty(A)) \simeq K_0(A)$

$$\frac{\mathcal{U}^\circ(A)}{D\mathcal{U}^\circ(A)} \xrightarrow{\simeq} \frac{\mathcal{U}^\circ_\infty(A)}{D\mathcal{U}^\circ_\infty(A)} \simeq \frac{\mathrm{Aff}\,T(A)}{\rho_A(K_0(A))}.$$

If $\Theta : \mathcal{U}^\circ(A) \rightarrow \mathcal{U}^\circ(B)$ is a hom, we then get a map

$$\begin{tikzcd}
\frac{U^0(A)}{DU^0(A)} \arrow[r, phantom, "\cong"] \arrow[d] & \frac{\mathrm{Aff}T(A)}{P_A(K_0(A))} \arrow[d] \arrow[r, phantom, "\circlearrowright"] & \mathrm{Aff}T(A) \arrow[l, two heads] \arrow[d, dotted, "\exists T_\theta"] \\
\frac{U^0(B)}{DU^0(B)} \arrow[r, phantom, "\cong"] & \frac{\mathrm{Aff}T(B)}{P_A(K_0(B))} & \mathrm{Aff}T(B) \arrow[l, two heads]
\end{tikzcd}$$

Question: Can you lift this? $\quad a \in A_{sa}$

Yes! Essentially, if θ is cts, $(\theta(e^{2\pi i t a}))_{t\in\mathbb{R}}$ is a one-parameter fam. of unitaries.

Stone's theorem $\Longrightarrow$ $\theta(e^{2\pi i t a}) = e^{2\pi i t b}$ for some $b \in B_{sa}$.

as one here.

Define $S_\theta : A_{sa} \longrightarrow B_{sa}$ via $S_\theta(a) = b$

Let $tr_A : A_{sa} \longrightarrow \mathrm{Aff}T(A)$ $\qquad \hat{a}(\tau) = \tau(a)$.

$a \longmapsto \hat{a}$

Lemma: S_θ induces a map

$\tilde{S}_\theta : \mathrm{Aff}T(A) \longrightarrow \mathrm{Aff}T(B)$.

$$\begin{tikzcd}
\frac{U^0(A)}{DU^0(A)} \arrow[r, "\cong"] \arrow[d] & \frac{\mathrm{Aff}T(A)}{P_A(K_0(A))} \arrow[d, "T_\theta"] \\
\frac{U^0(B)}{DU^0(B)} \arrow[r, phantom, "\cong"] & \frac{\mathrm{Aff}T(B)}{P_A(K_0(B))}
\end{tikzcd}$$

Prop: this does lift the T_θ from above.

if A, B are "nice".

e.g. If $\theta : \mathbb{T} \to \mathbb{T} = U^0(\mathbb{C})$ is cts hom, $S_\theta(\mathbb{R}_+) = \mathbb{R}_-$

$\theta(z) = z^n$ for some n. If $n < 0$,

& $\tilde{S}_\theta$ is also negative.

Theorem: If $\theta : U^0(A) \longrightarrow U^0(B)$ is an isometric isomorphism,

$\pm\tilde{S}_\theta$ is unital, positive & isometric.

So $T(A) \cong T(B)$

$\tilde{S}_\theta$ satisfies, when $\pi_1(U^\circ(A)) \simeq \pi_1(U^\circ_\infty(A)) \simeq K_0(A)$

B

(e.g. $A \otimes Z \simeq A$)

Then

$$\begin{tikzcd}
K_0(A) \arrow[r, "P_A"] \arrow[d, "\pm\pi_1(\theta)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Aff}\,T(A) \arrow[d, "\pm\tilde{S}_\theta"] \\
K_0(B) \arrow[r] & \mathrm{Aff}\,T(A)
\end{tikzcd}$$

commutes.