Higher Kazhdan projections, ℓ^2-Betti numbers & the coarse Baum-Connes conjecture

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Kazhdan's property (T)

In the mid 60's, Kazhdan defined property (T) for locally compact groups and used it as a tool to demonstrate that a large class of lattices in higher rank lie groups are finitely generated. E.g. $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$.

Characterisation

A group G has property (T) iff there exists a projection $p \in \mathrm{C}^*_{\max} G$ whose image under any unitary representation $(\pi, \mathcal{H})$ of G is the orthogonal projection $\mathcal{H} \to \mathcal{H}^{\pi(G)}$ onto the fixed vectors.

- This projection is called *Kazhdan projection*.
- The Kazhdan projection is unique and non-zero inside $\mathrm{C}^*_{\max} G$.
- For an infinite group G, Kazhdan projection in $\mathrm{C}^*_{\mathrm{red}} G$ is always zero.
- Existence of this projection violates a certain method of proof for the Baum-Connes conjecture, and is a source of counterexamples.

Baum-Connes conjecture, 1982

Let G be a countable discrete group. The Baum-Connes conjecture claims that the homomorphism (assembly map)

$$\mu_r\colon K_*^G(\underline{E}G) \to K_*(\mathrm{C}^*_{\mathrm{red}}G) \qquad * = 0, 1$$

is an isomorphism.

G an infinite property (T) group: if Dirac-dual Dirac method works

$$\begin{array}{ccc} K_0^G(\underline{E}G) & \xrightarrow{\mu_r} & K_0(\mathrm{C}^*_{\mathrm{red}}G) \\ & \searrow^{\mu_m} & \uparrow \lambda \\ & & K_0(\mathrm{C}^*_{\max}G) \end{array}$$

- interferes with surjectivity of the assembly map

$\rightsquigarrow$ counterexamples to various versions of the Baum-Connes conjecture

Higher Kazhdan projection

For G finitely generated (hence of type F_1), fix $(\pi, \mathcal{H})$

$$p_0 \colon \mathcal{H} \to \mathcal{H}^{\pi(G)} \qquad \text{Kazhdan projection}$$

$\rightsquigarrow$ higher degrees, one may use the identification $\mathcal{H}^{\pi(G)} = H^0(G, \mathcal{H})$

Definition (Li-Nowak-P 2020)

Let G be a discrete group of type F_{n+1}. Let $(\pi, \mathcal{H})$ be a unitary representation of G. The higher Kazhdan projection in degree n associated with π is the orthogonal projection $p_n \colon \mathcal{H}^{\oplus k_n} \to \tilde{H}^n(G, \mathcal{H})$.

Remark

- It always lives in the matrices over the von Neumann algebra generated by $\pi(G)$.
- Assuming spectral gap for the higher Laplacian $\pi(\Delta_n)$, it lives in the matrices over the C^*-algebra generated by $\pi(G)$.

ℓ^2-Betti numbers

group von Neumann algebra $LG = \overline{\mathbb{C}G}^{SOT} \subseteq \mathcal{B}(\ell^2 G)$

ℓ^2-Betti number

$$\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) \in [0, \infty]$$

canonical trace $\tau \colon LG \to \mathbb{C}$ defined by $\tau(\sum_{\text{finite}} c_g g) = c_e$

$$\tilde{H}^n(G, \ell^2 G) = p_n(\ell^2 G^{\oplus k_n}) \qquad \text{right LG-module}$$

$$\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) = \dim_{LG} p_n(\ell^2 G^{\oplus k_n}) = (Tr \otimes \tau)(p_n)$$

Identifying higher Kazhdan projections

Proposition (Folklore)

Assume $\lambda(\Delta_n)$ *has spectral gap so that* p_n *belongs to* $\mathrm{M}_{k_n}(\mathrm{C}^*_{red}G)$. *Then we have that*

$$\beta^n_{(2)}(G) = \tau_*([p_n])$$

In particular

- *if* $[p_n] = 0$ *in* $K_0(\mathrm{C}^*_{red}G)$, *then* $\beta^n_{(2)}(G) = 0$
- *if* $[p_n] \in \mathbb{Z} \cdot [1]$, *then* $\beta^n_{(2)}(G) \in \mathbb{Z}$

Example

- $K_0(\mathrm{C}^*_{\mathrm{red}}\mathbb{F}_n) = \mathbb{Z} \cdot [1]$, and $\beta^1_{(2)}(\mathbb{F}_n) = n - 1 \rightsquigarrow [p_1] = (n-1)[1]$
- $\beta^1_{(2)}(PSL(2,\mathbb{Z})) = 1/6 \rightsquigarrow [p_1] \notin \mathbb{Z} \cdot [1]$

Coarse Baum-Connes conjecture

X discrete metric space with bounded geometry
$\mathcal{H}$ separable, infinite dimensional Hilbert space
$\mathbb{C}[X] \subseteq \mathcal{B}(\ell^2(X, \mathcal{H}))$: $*$-algebra of finite propagation operators with compact entries $T_{(x,y)}$.

$$\text{Roe algebra} \quad \mathrm{C}^*[X] = \overline{\mathbb{C}[X]} \subseteq \mathcal{B}(\ell^2(X, \mathcal{H}))$$

Coarse Baum-Connes conjecture, Roe, 1993

For all X with bounded geometry the coarse assembly map

$$\mu_\bullet : \mathrm{KX}_\bullet(X) \to \mathrm{K}_\bullet(\mathrm{C}^*[X]) \qquad \bullet = 0, 1$$

is an isomorphism.

Application to the coarse Baum-Connes conjecture

- $\beta^n(G) = \dim_{\mathbb{C}} H^n(G, \mathbb{C}) \in \mathbb{N}$
- $\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) \in [0, \infty]$

Theorem (Li-Nowak-P 2020)

Let G be an exact residually finite group of type F_{n+1}. Let $N = \{N_i\}_i$ be a filtration of finite index normal subgroups of G. Let $\pi = \oplus_i \lambda_i$. Assume that $\pi(\Delta_n)$ has a spectral gap such that p_n belongs to $\mathrm{M}_{k_n}(\mathrm{C}^*_N G)$. If the coarse Baum-Connes assembly map for the box space $Y = \coprod G/N_i$ of G is surjective, then

$$\beta^n_{(2)}(N_i) = \beta^n(N_i)$$

for sufficiently large i.

$\rightsquigarrow$ strategy to find counterexamples to the conjecture

$\rightsquigarrow$ consequences of surjectivity of the conjecture

Thanks for listening!