Anomalous symmetries of simple operator algebras

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Outline

1. v.N algebraic background (anomalous symmetries of $\mathcal{R}$)
2. C^*-results (joint work with Sam Evington)

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Symmetries of $\mathcal{R}$

$$\mathcal{R} = \overline{\bigotimes_{i \in \mathbb{N}} (M_{n_i}(\mathbb{C}), \mathrm{tr})}$$

Classical symmetries.

- Let G be a countable discrete group then G acts by outer automorphisms on $\mathcal{R}$ via bernoulli shifts $G \curvearrowright \otimes_G \mathcal{R} \cong \mathcal{R}$.
- Also for any finite dimensional representation $\rho : G \curvearrowright \mathbb{C}^n$ consider
$$\mathrm{Ad}(\rho)^{\otimes \infty} : G \curvearrowright \mathcal{R}.$$

Definition

Let $\phi, \psi \in \mathrm{Aut}(\mathcal{R})$. We say they are conjugate if there exists $\theta \in \mathrm{Aut}(\mathcal{R})$ s.t. $\theta\phi\theta^{-1} = \psi$. We say they are outer conjugate if there exists $\theta \in \mathrm{Out}(\mathcal{R}) \cong \mathrm{Aut}(\mathcal{R})/\mathrm{Inn}(R)$ s.t. $\theta\overline{\phi}\theta^{-1} = \overline{\psi}$.

Classification of classical symmetries

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Connes classified automorphisms ϕ of $\mathcal{R}$ up to outer conjugacy. The invariant being:

- The order $n \in \mathbb{N} \cup \{\infty\}$ of $\overline{\phi}$ in $\mathrm{Out}(\mathcal{R})$.
- If $\phi^n = \mathrm{Ad}(u)$, the n^{th} root of unity ω such that $\phi(u) = \omega u$.

The n^{th} root of unity

it follows that

$$\phi^{n+1} = \phi\phi^n = \phi \mathrm{Ad}(u) = \mathrm{Ad}(\phi(u))\phi$$

and

$$\phi^{n+1} = \phi^n \phi = \mathrm{Ad}(u)\phi$$

So $\phi(u) = \omega u$ for some ω in $Z(\mathcal{R}) = \mathbb{C}$. Moreover, ω is an n^{th} root of unity.

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How Connnes constructs these:

For each (n, ω) Connes uses $n \times n$ "clock and shift" matrices to builds a sequence of inner automorphisms $\alpha_k^{(n,\omega)}$ on $\mathcal{R}$ which is pointwise eventually constant on $\bigodot_{i \in \mathbb{N}} M_n$. So it converges pointwise to an automorphism s_n^ω on $\mathcal{R}$. The sequence is made such that $(s_n^\omega)^n = \mathrm{Ad}(u_\omega)$ and $s_n^\omega(u_\omega) = \omega u_\omega$.

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Connes construction also builds (n, ω) automorphisms on $M_{n^\infty} = \overline{\bigotimes M_n}^{\|\cdot\|}$.

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The associated class in H^3

Let $\overline{\theta}$ be a G-kernel on A then picking lifts θ_g for each $g \in G$ one has unitaries s.t. $\mathrm{Ad}(u_{g,h})\theta_g\theta_h = \theta_{gh}$:

$$\theta_{g(hk)} = \mathrm{Ad}(u_{g,hk}\theta_g(u_{h,k}))\theta_g\theta_h\theta_k$$

$$\theta_{(gh)k} = \mathrm{Ad}(u_{gh,k}u_{g,h})\theta_g\theta_h\theta_k$$

So there exists $\omega(g,h,k) \in Z(U(A)) = \mathbb{C}$ such that $\omega(g,h,k)$ is the multiplicative difference between these two unitaries.

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We are interested in factors or simple We call a lift $\{u, \theta\}$ of a G-kernel with associated 3-cocycle $\omega \in Z^3(G, \mathbb{T})$ an ω-anomalous action.

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Explicit representatives are given in terms of the n^{th} roots of unity.

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As part of Jones classification of group actions he shows the existence of G kernels for any countable discrete group G and $\omega \in H^3(G, \mathbb{T})$ on $\mathcal{R}$. He classifies injective G-kernels for G finite up to outer conjugacy by their class in H^3.

How does the existence of ω-anomalous actions translate in the case of C^*-analogues of $\mathcal{R}$? There is nothing like the "trivial action" for non-trivial ω.

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What C^*-algebras are we interested in?

We are interested in the existence of ω-anomalous actions on a particularly nice class of C^*-algebras which can be classified by K-theoretic and tracial data. We will call these “classifiable”.

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The Jiang-Su algebra $\mathcal{Z}$

- simple, infinite dimensional and has no non-trivial projections.
- The K theory is the same as that of the complex numbers. $(K_0(\mathcal{Z}), K_0(\mathcal{Z})^+, [1_{\mathcal{Z}}]) = (\mathbb{Z}, \mathbb{Z}^+, 1)$, $K_1(\mathcal{Z}) = 0$.
- has a unique trace.
- $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$ in a particularly nice way. (in particular $\mathcal{Z} \cong \otimes_{n=1}^{\infty} \mathcal{Z}$).

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We are interested in the existence of ω-anomalous actions on a particularly nice class of C^*-algebras which can be classified by K-theoretic and tracial data. We will call these "classifiable".

Note

"Classifiable" C^*-algebras are $\mathcal{Z}$-absorbing ($A \otimes \mathcal{Z} \cong A$). **Any countable discrete group acts through outer automorphisms on any classifiable C^*-algebra.**

Theorem 1 (Evington-G)

Let G be a group. Suppose there exists an ω-anomalous G action on $\mathcal{Z}$. Then $[\omega] = 0$ in $H^3(G, \mathbb{T})$.

C^* obstruction results

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Theorem 2 (Evington-G)

Let G be a group. Suppose there is an ω-anomalous G action on the the UHF algebra $\bigotimes_{k\in\mathbb{N}} M_{n_k}$. Suppose ω has finite order, let r be the order of $[\omega]$ in $H^3(G, \mathbb{T})$. Then r divides $|G|$ and r^∞ divides $\prod_{k\in\mathbb{N}} n_k$.

The obstructions arises from the unitary algebraic K_1 group which is defined as

$$K_1^{\mathrm{alg}}(A) = \frac{U_\infty(A)}{[U_\infty(A), U_\infty(A)]}.$$

This is a finer equivalence relation than homotopy i.e. we always have a surjection

$$K_1^{\mathrm{alg}}(A) \twoheadrightarrow K_1(A)$$

$\mathrm{K}_1^{\mathrm{alg}}(\mathbb{C})$

$K_1^{\mathrm{alg}}(\mathbb{C}) \cong \mathbb{T}$ and the isomorphism is induced by the determinant $\det : U_\infty(\mathbb{C}) \to \mathbb{T}$ whose kernel is $[U_\infty(\mathbb{C}), U_\infty(\mathbb{C})]$.

This can be calculated for "classifiable" C*-algebras through the Skandalis de la Harpe determinant. For example

$$\begin{aligned} K_1^{\mathrm{alg}}(\mathcal{Z}) &\cong \mathbb{R}/\mathbb{Z} \\ [e^{2\pi i h}] &\underset{\Delta_{\mathcal{Z}}}{\to} \tau(h) + \mathbb{Z} \end{aligned}$$

$$K_1^{\mathrm{alg}}(\mathcal{Z}) \cong \mathbb{R}/\mathbb{Z}$$
$$[e^{2\pi i h}] \underset{\Delta_{\mathcal{Z}}}{\rightarrow} \tau(h) + \mathbb{Z}$$

The cyclic case

Suppose you have a unitary $u \in \mathcal{Z}$ such that $\theta(u) = \omega u$ then applying K_1^{alg} you get

$$K_1^{\mathrm{alg}}(\theta(u)) = K_1^{\mathrm{alg}}(\omega) + K_1^{\mathrm{alg}}(u)$$

So $K_1^{\mathrm{alg}}(\omega) = 0$ i.e. $\omega = 1$.

Algebraic K_1 obstruction

- $K_1^{\text{alg}}(M_{n^\infty}) = \frac{\mathbb{R}}{\mathbb{Z}[\frac{1}{n}]}$.
- $K_1^{\text{alg}}(\mathcal{R}) = 0$ **no obstruction**.

Algebraic K_1 obstruction

We can do things in more generality

- The results are not just ad hoc. We have a general obstruction arising from K_1^{alg}.
- We can also achieve existence results for these actions e.g. on UHF for arbitrary finite G and some C^*-algebras with non-trivial K_1 (e.g. Bunce Deddens, $\mathcal{A}_\theta$).

Questions?