General Stable Rank of C*-algebras

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Overview

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- $C(X)$-algebras

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- The general stable rank is homotopy invariant, that is, if there are two homotopically equivalent C^*-algebras A and B, then $gsr(A) = gsr(B)$.

Projective Module

A finitely generated A-module P is said to be projective if there exists another A-module Q such that

$$P \oplus Q \cong A^n \text{ for some } n.$$

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then when can we conclude that, P is free,

$$P \cong A^{n-1}?$$

General Stable Rank

Definition

Given a C^*-algebra A, the *general stable rank* of A ($gsr(A)$) is the least natural number $n \geq 1$ such that for any projective A-module P,

$$P \oplus A \cong A^m \implies P \cong A^{m-1}$$

for all $m \geq n$.

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Assumption

We will assume, throughout the talk, that A has the *Invariant Basis Number property*, that is,

$$A^m \cong A^n \implies m = n.$$

Unimodular Vectors

Given a unital C^*-algebra A. For each $n \in \mathbb{N}$, we define

$$Lg_n(A) := \{(a_i) \in A^n \text{ such that } \exists\ (b_i) \in A^n \text{ and } \sum_{i=1}^{n} b_i a_i = 1_A\}.$$

Note, here that $e_i = (0, \cdots, 1, \cdots, 0) \in Lg_n(A)$.

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Conversely, if P is a finitely generated projective A-module such that

$$f\colon P \oplus A \cong A^n.$$

Then define $Q = f(P \oplus 0)$ and $\underline{a} = f(0, 1)$. We observe that

$$Q \cong P \text{ and } Q \oplus A\underline{a} = A^n.$$

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translates to answer the following question:

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if and only if there is a $T \in GL_m(A)$ such that $T\underline{a} = e_m$.

Equivalent definition of gsr

Definition

The *general stable rank (gsr)* of A is also defined as the least integer $n \geq 1$, such that

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Here, the action of $GL_m(A)$ on $Lg_m(A)$ is defined by the usual matrix multiplication.

for non-unital C^*-algebras

If A is a non-unital C^*-algebra then the general stable rank of A is defined as the general stable rank of the unitization, $\tilde{A}$, of A.

From now on, we assume that all C^*-algebras are unital.

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- $gsr(A) \leq tsr(A) + 1$.

Examples

- Consider the C^*-algebra $\mathbb{C}$. Since every projective $\mathbb{C}$-module P is a $\mathbb{C}$-vector space, P has a basis and hence $P \cong \mathbb{C}^n$ for some n. Thus
$$gsr(\mathbb{C}) = 1.$$

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- $gsr(B(\mathcal{H})) = \infty$ if $\mathcal{H}$ is an infinite dimensional Hilbert space.

- $gsr(C(X)) \leq \left\lceil \frac{dim(X)}{2} \right\rceil + 1.$

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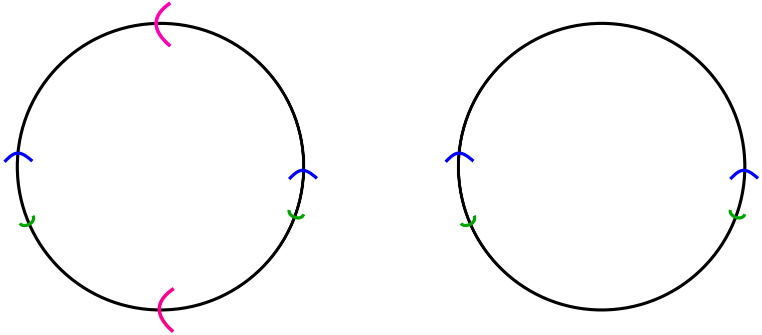
- If $gsr(A) = 1$, then A has cancellation.

Covering Dimension

Consider the unit circle $\mathbb{T}$ with an open cover as in, the left circle, figure. Then we can choose the refinement as in the right circle. Therefore, any point of the unit circle $\mathbb{T}$ lies in at most two of the sets which cover $\mathbb{T}$.



$C(X)$-algebra

Definition

For a compact Hausdorff space X, a C^*-algebra A is said to be a $C(X)$-algebra if there is a unital $*$-homomorphism

$$\phi \colon C(X) \to Z(A)$$

where $Z(A)$ is the center of the C^*-algebra A.

A $C(X)$-algebra as a bundle

- Let $x \in X$ be a fixed point, $I_x := \{f \in C(X) : f(x) = 0\}$ is an ideal in $C(X)$.

$$A(x) := A/I_xA \text{ is again a } \mathrm{C}^*\text{-algebra.}$$

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$$A(x) := A/I_xA \text{ is again a } \mathrm{C}^*\text{-algebra.}$$

- Then, we get the natural quotient map

$$A \longrightarrow A(x), \; a \mapsto a(x).$$

For each $a \in A$, we define the map

$$X \to \mathbb{R}, \; x \mapsto \|a(x)\|.$$

It turns out the above map is always upper semicontinuous.

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- $gsr(A)$ must depend on $dim(X)$, $gsr(A(x))$ and "some" numbers which again depend on $dim(X)$.

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Let A be a $C(X)$-algebra where X is a compact Hausdorff topological space of dimension zero. Then

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Let A be a $C(X)$-algebra where X is a compact Hausdorff topological space of dimension zero. Then

- $$gsr(A) \leq \sup\{gsr(A(x)) : x \in X\}.$$

Theorem (—— , Vaidyanathan)

Let A be a $C(X)$-algebra where X is an n-dimensional compact metric space and $A(x)$'s are fibres of A such that $K_1(A(x)) = 0$ for each $x \in X$. Then

$$gsr(A) \leq \sup\{gsr(C(\mathbb{T}^n) \otimes A(x)) : x \in X\}.$$

Crossed product C^*-algebra

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- Let $\alpha \colon G \to Aut(A)$ be an action.
- We can associate a C^*-algebra to the above system called the crossed product C^*-algebra and is denoted by $A \rtimes_\alpha G$.
- To each $g \in G$, associate the function $u_g \in C(G, A, \alpha)$ where $u_g(h) = 1$ iff $h = g$. Then $A \rtimes_\alpha G$ is the universal C^*-algebra generated by A and u_g subject to the relation $u_g a u_g^* = \alpha_g(a)$ for all $g \in G$.

Questions

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- Can we put some restrictions on the action α to get "nice" bounds?

Rokhlin property

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- $\sum_{g \in G} e_g = 1$
- $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$
- $\|e_g a - a e_g\| < \epsilon$ for all $g \in G, a \in F$.

Examples

- (A, G, id), the trivial action does not have Rokhlin property since $id_g = Id_A$ for all $g \in G$ then

$$id_{g^{-1}}(e_g) = e_g \neq e_{g_0} = e_{g^{-1}g}$$

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- Let B be any C^*-algebra. Then the left translation action

$$\alpha\colon G \longrightarrow Aut(C(G, B)) \text{ given by}$$

$$s \mapsto \alpha_s \text{ where } \alpha_s(f)(x) = f(s^{-1}x) \text{ for } f \in C(G, B).$$

For a given finite set $F \subset C(G, B)$ and a given $\epsilon > 0$, the following set of mutually orthogonal projections

$$e_g(h) = \begin{cases} 1_B, & \text{if } g = h \\ 0, & \text{otherwise} \end{cases}$$

works.

Theorem (——, Vaidyanathan)

Let $\alpha : G \to Aut(A)$ be an action of a finite group G on a separable, unital C^*-algebra A with the Rokhlin property. Then

- $$gsr(A \rtimes_\alpha G) \leq \left\lceil \frac{gsr(A) - 1}{|G|} \right\rceil + 1.$$

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Let $\alpha : G \to Aut(A)$ be an action of a finite group G on a separable, unital C^*-algebra A with the Rokhlin property. Then

- $$gsr(A \rtimes_\alpha G) \leq \left\lceil \frac{gsr(A) - 1}{|G|} \right\rceil + 1.$$

- *In particular,$gsr(A) = 1$, then the same is true for $gsr(A \rtimes_\alpha G) = 1$.*

Future Directions

Similar questions for

- finite group and actions with tracial Rokhlin property and other higher dimensional Rokhlin properties (called Rokhlin diemension),
- non-finite compact groups and actions with Rokhlin property.

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Thank You!