# The Connes Embedding Problem: from operator algebras to groups and quantum information theory

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**Definition** (Anantharaman-Delaroche '05): A unital quantum channel $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *factorizable* if $\exists$ vN alg $(N, \psi)$ with n.f. tracial state and unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$ : $T = \beta^* \circ \alpha$.

$$\begin{tikzcd}
M_n(\mathbb{C}) \arrow[rr, "T"] \arrow[dr, "\alpha"'] & & M_n(\mathbb{C}) \arrow[dl, "\beta"'] \\
& M_n(\mathbb{C}) \otimes N \arrow[ur, bend right, "\beta^* = \beta^{-1} \circ \mathbb{E}_{\beta(M_n(\mathbb{C}))}"'] &
\end{tikzcd}$$

**Theorem** (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel iff $\exists$ $(N, \tau_N)$ finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

- (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C$^*$-algebras):

- $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^*\,\iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the *canonical inclusions*. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some c.p. lin map $T_\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, which turns out to be a factoriz quantum channel!

In fact, the map $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

$$\Phi(T_{\mathrm{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\mathrm{fin}}(n),$$

where $T_{\mathrm{fin}}$ = tracial states that factor through fin. dim. $C^*$-alg.

The affine cont surj $\Phi\colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto T_\tau$, satisfies

- $\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n)$,
- $\Phi(\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}) = \overline{\mathcal{FM}_{\rm fin}(n)}$,

where $T_{\rm fin}$ = tracial states that factor through fin. dim. $C^*$-alg.

**Recall**: CEP positive answer $\iff$ $\mathcal{FM}(n) = \overline{\mathcal{FM}_{\rm fin}(n)}$, $\forall n \geq 3$.

**Question**: What can we say about $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}$?

- (Exel–Loring '92): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ residually finite dim. (RFD)
- (Blackadar '85): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ semi-projective.

In general, given $A =$ unital $C^*$-algebra, we have inclusions:

$$T_{\mathrm{fin}}(A) \subseteq \overline{T_{\mathrm{fin}}(A)} \subseteq T_{\mathrm{qd}}(A) \subseteq T_{\mathrm{am}}(A) \subseteq T_{\mathrm{hyp}}(A) \subseteq T(A),$$

where $T_{\mathrm{qd}}(A) =$ quasi-diagonal traces, $T_{\mathrm{am}}(A) =$ amenable (=liftable) traces, $T_{\mathrm{hyp}}(A) =$ hyperlinear traces (i.e., traces $\tau$ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

▶ If $A$ is *separable*, then $\overline{T_{\mathrm{fin}}(A)}$, $T_{\mathrm{qd}}(A)$, $T_{\mathrm{am}}(A)$, resp., $T_{\mathrm{hyp}}(A)$ contains a faithful trace iff $A$ is RFD, quasi-diagonal, embeds into $\mathcal{R}^\omega$ with ucp lift to $\ell^\infty(\mathcal{R})$, resp., embeds into $\mathcal{R}^\omega$.

- CEP pos answer **iff** $T_{\mathrm{hyp}}(A) = T(A)$, for all $C^*$-alg $A$.
- It is open whether $T_{\mathrm{qd}}(A) = T_{\mathrm{am}}(A)$. There are strong positive results!
- (N. Brown '06): $\exists$ exact RFD $C^*$-alg $A$ s.t. $T_{\mathrm{am}}(A) \neq T_{\mathrm{hyp}}(A)$.
- $A$ (weakly) semi-projective $\implies$ $\overline{T_{\mathrm{fin}}(A)} = T_{\mathrm{qd}}(A)$
- (Hadwin–Shulman '17): $\exists$ RFD $C^*$-alg $A$ s.t. $\overline{T_{\mathrm{fin}}(A)} \neq T_{\mathrm{qd}}(A)$.

**Thm** (Rørdam-M '20): $\overline{T_{\mathrm{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\mathrm{hyp}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

**Thm** (Rørdam-M): $\overline{T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))} = T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

**Cor:** CEP pos **iff** $T_{\rm hyp}(M_n(\mathbb{C}) * M_n(\mathbb{C})) = T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, $\forall\, n \geq 3$.

**Further results**: Let $A$ be a unital $C^*$-algebra.

- If $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, then $A$ is gen by $n^2$ elem.
- If $A$ is gen by $n-1$ elem, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$.

**Theorem**: Each metrizable Choquet simplex is affinely homeo to a face of $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

**Question**: Is $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$ the Poulsen simplex?

## Groups, $C^*$-tensor norms, Tsirelson's conjecture and CEP

- Let $\Gamma = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$ ($n$ free factors), $n, k \geq 2$.

**Theorem** (Fritz/Junge et. al. '09):

- $\mathcal{C}_{qa}(n,k) = \left\{ \left[ \varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{C}_{qc}(n,k) = \left\{ \left[ \varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

- $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ is RFD [$\Rightarrow \mathcal{C}_{qs}^{\text{fin}}(n,k) \overset{\text{dense}}{\subseteq} \mathcal{C}_{qs}(n,k)$].

The Thm above proves "(i) $\Rightarrow$ (iv)" below:

**Theorem** (Kirchberg '93, Fritz/Junge et. al. '09, Ozawa '12): TFAE:

(i) $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$, $\forall n, k \geq 2$,

(ii) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,

(iii) Connes embedding problem has positive answer,

(iv) Tsirelson's conjecture is true, i.e., $C_{qa}(n,k) = C_{qc}(n,k)$, $\forall n, k \geq 2$.

Ozawa proved (iv) $\Longrightarrow$ (i).

## A new approach, via analysis of *synchronous corellations*

Revisited notation: $\Gamma = \mathbb{F}(n,k) = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$, $n$ copies, $n, k \geq 2$.

- $$\begin{aligned} C^*(\mathbb{Z}_k) &= C^*(u \mid uu^* = u^*u = 1,\ u^k = 1) \\ &= C^*(q_1, \ldots, q_k \mid q_j = q_j^* = q_j^2,\ \sum_{j=1}^{k} q_j = 1). \end{aligned}$$
- $$C^*(\mathbb{F}(n,k)) = C^*(q_{j,x} \mid q_{j,x} = q_{j,x}^* = q_{j,x}^2,\ \textstyle\sum_{j=1}^{k} q_{j,x} = 1).$$

**Definition**: A "correlation" $[(p(i,j \mid x,y)]$ is *synchronous* if whenever $i \neq j$, $p(i,j \mid x,x) = 0$, $\forall\, 1 \leq x \leq n$.

**Theorem** (PSSTW '16): We have the following identities of *synchronous* correlation matrices:

$$\begin{aligned} C_{qc}^s(n,k) &= \left\{ \left[\tau(q_{j,x}q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T(C^*(\mathbb{F}(n,k)))\right\} \\ C_q^s(n,k) &= \left\{ \left[\tau(q_{j,x}q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T_{\text{fin}}(C^*(\mathbb{F}(n,k)))\right\}. \end{aligned}$$

- Consequently, we deduce:

$$C_{qa}^s(n,k) = \left\{ \left[\tau(q_{j,x}q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in \overline{T_{\text{fin}}(C^*(\mathbb{F}(n,k)))}\right\}.$$

**Theorem** (Kim-Paulsen-Schafhauser '17, Ozawa '12): TFAE

(1) Connes embedding problem has positive answer.

(2) $C^s_{qa}(n,k) = C^s_{qc}(n,k)$, $\forall n,k \geq 2$.

(3) Tsirelson's conjecture is true, i.e., $C_{qa}(n,k) = C_{qc}(n,k)$, $\forall n,k \geq 2$.

**Note**: • (3) $\Longrightarrow$ (1) was shown by Ozawa, using Kirchberg's Thm that "CEP pos. answer **iff** $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$."

• (3) $\Longrightarrow$ (2) is trivial.

• [KPS] proved (1) $\Longleftrightarrow$ (2) using different reformulations of CEP.

Next, I'd like to discuss a proof (with M. Rørdam)of (1) $\Longrightarrow$ (2), based on arguments of C. Schafhauser in a recent talk (AIM, May 2021).

**Proposition** (based on Schafhauser): If CEP has a positive answer, then

$$\overline{T_{\text{fin}}(C^*(\mathbb{F}(n,k)))} = T(C^*(\mathbb{F}(n,k))).$$

We'll need a few intermediate results, namely:

**Lemma** (Folklore): Let $I \lhd M$, where $M =$ unital $C^*$-alg of real rank zero (e.g., $M$ a vN algebra), and let $\pi\colon M \to M/I$ be the quotient mapping.
If $q_1, \dots, q_k \in M/I$ are projections s.t. $\sum_{j=1}^k q_j = 1$, then
$\exists\, p_1, \dots, p_k \in M$ projections with $\sum_{j=1}^k p_j = 1$ and $\pi(p_j) = q_j$.

**Corollary:** Let $I \lhd M$, $\pi\colon M \to M/I$ as above. Then each unital $^*$-hom $\varphi\colon C^*(\mathbb{F}(n,k)) \to M/I$ lifts to a unital $^*$-hom $\psi\colon C^*(\mathbb{F}(n,k)) \to M$ s.t.

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ C^*(\mathbb{F}(n,k)) \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi"'] & M/I \end{tikzcd}$$

**Reformulation of CEP:** For all sep. unital tracial $C^*$-algs $(A, \tau)$, there is a unital trace- preserving $^*$-hom $\varphi\colon A \to \prod_{n=1}^{\infty} M_{k_n}/I^{\omega}$, for some $k_n \geq 1$.

- By GNS we have unital trace preserving $^*$-hom $(A, \tau) \to (\pi_\tau(A)'', \overline{\tau})$, and $(\pi_\tau(A)'', \overline{\tau})$ is a finite von Neumann algebra with n.f.t.s. $\overline{\tau}$.

- Connes' "matricial microstate" (re)formulation of CEP implies that each sep. finite von Neumann algebra $(M, \tau)$ with n.f.t.s. $\tau$ admits a trace preserving unital embedding $M \to \prod_{n=1}^{\infty} M_{k_n}/I^{\omega}$.

**Proof of Prop**: Assume CEP holds. Let $\tau \in T(C^*(\mathbb{F}(n,k)))$. Then $\exists$:

$$\begin{tikzcd}
 & \prod_{n=1}^{\infty} M_{k_n} \arrow[d, "\pi"] \\
C^*(\mathbb{F}(n,k)) \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi"'] & \prod_{n=1}^{\infty} M_{k_n}/I^{\omega}
\end{tikzcd}$$

s.t. $\tau = \tau_\omega \circ \varphi$. The lift $\psi$ exists by the previous corollary.

Assume CEP has pos. answer. Let $\tau \in T(C^*(\mathbb{F}(n,k)))$. Then $\exists$:

$$\begin{tikzcd} & \prod_{n=1}^{\infty} M_{k_n} \arrow[d, "\pi"] \\ C^*(\mathbb{F}(n,k)) \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi"'] & \prod_{n=1}^{\infty} M_{k_n}/I^{\omega} \end{tikzcd}$$

s.t. $\tau = \tau_\omega \circ \varphi$. The lift $\psi$ exists by the previous corollary.

Write $\psi = (\psi_n)_{n\geq 1}$ with $\psi_n \colon C^*(\mathbb{F}(n,k)) \to M_{k_n}$ unital $^*$-homs.

By definition of $\tau_\omega$, for all $a \in C^*(\mathbb{F}(n,k))$ we have

$$\tau(a) = (\tau_\omega \circ \varphi)(a) = \lim_{n\to\omega} (\mathrm{tr}_{k_n} \circ \psi_n)(a)$$

and $\mathrm{tr}_{k_n} \circ \psi_n \in T_{\mathrm{fin}}(C^*(\mathbb{F}(n,k)))$, which proves $\tau \in \overline{T_{\mathrm{fin}}(C^*(\mathbb{F}(n,k)))}$. $\square$

- Further, use the Paulsen-Severini-Stahlke-Todorov-Winter '16 theorem, to conclude that (1) $\implies$ (2) in the Kim-Paulsen-Schafhauser theorem.

## Groups, $C^*$-algebras, Tsirelson's Conjecture, Complexity and **CEP**

**Theorem** (Kirchberg '93, Fritz/Junge et al '09, Ozawa '12): TFAE:

(i) $C^*(\mathbb{F}_{n,k}) \otimes_{\max} C^*(\mathbb{F}_{n,k}) = C^*(\mathbb{F}_{n,k}) \otimes_{\min} C^*(\mathbb{F}_{n,k})$, $\forall n, k \geq 2$.

(ii) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$.

(iii) The Connes Embedding Problem has a positive answer.

(iv) Tsirelson's Conjecture is true: $\mathrm{cl}(\mathcal{C}_{qs}(n,k)) = \mathcal{C}_{qc}(n,k)$, $\forall n, k \geq 2$.

**Posted on arXiv, Jan. 13, 2020:** $MIP^* = RE$, Ji, Natarajan, Vidick, Wright, Yuen, 165 pp.

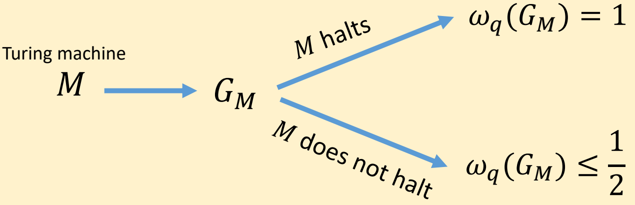
Proving that the complexity class MIP* (quantum version of complexity class MIP=languages with a Multiprover Interactive Proof) contains an undecidable language, they conclude that Tsirelson's Conjecture is **false**!

- New version (with corrections) 206 pp., posted on arXiv, Sept. 29, 2020.

*The last two slides are from Henry Yuen's online lecture at Univ. Texas, Austin (March '20).*

# MIP* = RE

**<u>Main result</u>** There exists an computable map $M \mapsto G_M$ from Turing machines to nonlocal games such that




**½ can be replaced by any constant less than 1.**

# Implications

- Turing 1936: No algorithm can solve the Halting Problem.
- Thus there is no algorithm to approximate $\omega_q \pm \epsilon$ for any $\epsilon$, and in particular the Search Above/Search Below algorithm cannot converge for all $G$
- Thus there exists a game $G$ such that $\omega_q(G) \neq \omega_{qc}(G)$.
- This implies negative answer to Tsirelson's problem: $C_{qa} \neq C_{qc}$
- Therefore Connes' embedding conjecture is false.