The Connes Embedding Problem: from operator algebras to groups and quantum information theory

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- Let $\Gamma = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$ (n free factors), $n, k \geq 2$.

Theorem (Fritz/Junge et. al. '09):

- $\mathcal{C}_{qa}(n,k) = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{C}_{qc}(n,k) = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

- $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ is RFD $[\Rightarrow \mathcal{C}_{qs}^{\text{fin}}(n,k) \overset{\text{dense}}{\subseteq} \mathcal{C}_{qs}(n,k)]$.

The Thm above proves "(i) $\Rightarrow$ (iv)" below.

Theorem (Kirchberg '93, Fritz/Junge et. al. '09, Ozawa '12): TFAE:

(i) $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ for all $n, k \geq 2$,

(ii) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,

(iii) Connes embedding problem has positive answer,

(iv) Tsirelson's conjecture is true.

A tale of two C^*-algebras: $C^*(\mathbb{F}_\infty)$ and $\mathcal{B}(H)$

Theorem (Kirchberg '93): The Connes Embedding Problem has positive answer **iff** $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.

Furthermore, Kirchberg proved that

$$C^*(\mathbb{F}_\infty) \otimes_{\min} \mathcal{B}(H) = C^*(\mathbb{F}_\infty) \otimes_{\max} \mathcal{B}(H)$$

and asked whether

$$\mathcal{B}(H) \otimes_{\min} \mathcal{B}(H) = \mathcal{B}(H) \otimes_{\max} \mathcal{B}(H).$$

- The latter was later answered in the negative by Junge-Pisier '95, using a non-commutative version of Grothendieck's inequality. Pisier-Ozawa '14 showed that there are at least $2^{\aleph_0}$ non-equivalent norms on $\mathcal{B}(H) \otimes \mathcal{B}(H)$.

A tale of two properties: WEP and L(LP)

Definition (Lance): A C^*-alg A has the *weak expectation property* (WEP) if $\exists$ repn $\pi\colon A \to B(H)$ and c.c.p. map $\Phi\colon B(H) \to A^{**}$ s.t. $(\Phi \circ \pi)(a) = a$, for $a \in A$. (Here $A^{**} = (A^*)^*$ is the double dual of A, with A viewed as a Banach space. Note that A^{**} is a vN alg.)

- Defin is indep of the faithful repn. If A unital, may choose π, Φ unital.

Example: $B(H)$ has (WEP). In fact, any injective vN alg M has (WEP).
Recall: $M \subseteq B(H)$ is injective if $\exists$ conditional expectation $E\colon B(H) \to M$.
By Connes '76, M injective **iff** M hyperfinite.

Theorem: A C^*-alg A is nuclear **iff** A^{**} is an injective vN alg.

Corollary: All nuclear C^*-algebras have (WEP).

Theorem: Let Γ be a countable inf group. Then $C^*_{\text{red}}(\Gamma)$ has (WEP) **iff** $C^*_{\text{red}}(\Gamma)$ nuclear **iff** Γ amenable **iff** $C^*(\Gamma) = C^*_{\text{red}}(\Gamma)$.

Corollary: $C^*_{\text{red}}(\mathbb{F}_n)$ not (WEP), $2 \leq n \leq \infty$.

Proposition: Having (WEP) does not pass to sub-C^*-algebras.

Proof: By Kirchberg's Thm, every separable exact C^*-algebra embeds into the Cuntz algebra $\mathcal{O}_2$, which is nuclear, hence (WEP). But $C^*_{\text{red}}(\mathbb{F}_n)$ not WEP, $n \geq 2$.

Definition: A C^*-alg which is a *quotient of a C^*-alg with (WEP)* is said to have (QWEP).

- All C^*-algs with (WEP) are (QWEP). In particular, nuclear C^*-algs are (QWEP). Also, any quotient of a C^*-alg with (QWEP) is again (QWEP).

Conjecture (Kirchberg '93): All C^*-algebras are (QWEP).

- Kirchberg's (QWEP) conj. holds $\Leftrightarrow$ $C^*(\mathbb{F}_\infty)$ is (QWEP).

Proposition: $C^*(\mathbb{F}_\infty)$ is (QWEP) $\iff$ $C^*(\mathbb{F}_\infty)$ is (WEP).

Proposition: $\mathcal{R}^\omega$ is (QWEP), and so are all finite vN algs that embed into $\mathcal{R}^\omega$.

Definition: Let $\mathcal{I} \lhd B$ and A be C^*-algs. A c.c.p. map $\varphi\colon A \to B/\mathcal{I}$ is *ccp-liftable* if $\exists$ ccp map $\psi\colon A \to B$ st $\varphi = \pi \circ \psi$:

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ A \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi"'] & B/\mathcal{I} \end{tikzcd}$$

If $\forall$ finite dim. op. system $E \subseteq A$ there exists ccp map $\psi\colon E \to B$ st $\varphi|_E = \pi \circ \psi$, then φ is *locally ccp-liftable*.

A has (LLP) if all ccp maps $\varphi\colon A \to B/\mathcal{I}$ are locally ccp-liftable.
Respectively, A has (LP) if all ccp maps $\varphi\colon A \to B/\mathcal{I}$ are ccp-liftable.

Note: (LP) implies (LLP). Converse is open.

- By (Choi-Effros '76): All separable nuclear C^*-algebras have (LP), since all ccp maps from or into a nuclear C^*-alg are nuclear. In particular, $C^*(\Gamma)$ has (LP), when Γ ctble amenable.

Theorem (Kirchberg '93): $C^*(\mathbb{F}_n)$ has (LP), for $2 \leq n \leq \infty$.

- (Ioana-Spaas-Wiersma '21): Let $\Gamma = SL_n(\mathbb{Z})$, $n \geq 3$. Then $C^*(\Gamma)$ fails (LLP), hence it fails (LP).

Theorem (Kirchberg): $C^*(\mathbb{F}_\infty) \otimes_{\max} B(H) = C^*(\mathbb{F}_\infty) \otimes_{\min} B(H)$.

Theorem (Kirchberg): Let A, B be C^*-algs. Then:

(i) A has (LLP) $\iff$ $A \otimes_{\max} B(H) = A \otimes_{\min} B(H)$.

(ii) B has (WEP) $\iff$ $C^*(\mathbb{F}_\infty) \otimes_{\max} B = C^*(\mathbb{F}_\infty) \otimes_{\min} B$.

(iii) If A has (LLP) and B has (WEP), then $A \otimes_{\max} B = A \otimes_{\min} B$.

Theorem (Kirchberg '93): TFAE:

(i) All (separable) C^*-algs are (QWEP),

(ii) $C^*(\mathbb{F}_\infty)$ is (QWEP),

(iii) (LLP) $\implies$ (WEP) for all C^*-algs,

(iv) $C^*(\mathbb{F}_\infty)$ is (WEP),

(v) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,

(vi) CEP has a positive answer,

(vii) $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty)$ has a faithful trace,

(viii) $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty)$ is RFD (residually finite dimensional).

Note: $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty) \cong C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.

- (Choi '80): $C^*(\mathbb{F}_\infty)$ is RFD, hence has a faithful trace. Furthermore, being RFD is preserved by $\otimes_{\min}$. (Brown-Ozawa): $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty)$ is QD.

Proposition (Kirchberg): Let A, B unital C^*-algs. If $A \otimes_{\max} B$ has a faithful trace, then $A \otimes_{\max} B = A \otimes_{\min} B$.

Now onto quantum channels:

- (Choi '73): $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear is completely positive (cp) iff

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1\,, \ldots\,, a_d \in M_n(\mathbb{C})$ can be chosen linearly independent.

- The Choi matrix C_T of a linear map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is

$$C_T = \sum_{i,j=1}^{n} e_{ij} \otimes T(e_{ij}) \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) = M_{n^2}(\mathbb{C}),$$

where $\{e_{ij}\}_{1\leq i,j\leq n}$ are matrix units for $M_n(\mathbb{C})$. Then

$$T(e_{ij}) = \sum_{k,\ell=1}^{n} C_T(i,j;k,\ell)\, e_{k\ell}, \qquad 1 \leq i,j \leq n,$$

where $C_T(i,j;k,\ell) = \langle C_T, e_{ij} \otimes e_{k\ell} \rangle_{\mathrm{Tr}_n \otimes \mathrm{Tr}_n}$ (matrix coefficients).

- (Choi '75): T completely positive **iff** C_T positive matrix.

A cp trace-preserving map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a quantum channel.

Examples of (unital) quantum channels:

- Automorphisms of $M_n(\mathbb{C})$: $T \in \mathrm{Aut}(M_n(\mathbb{C}))$ iff $\exists u \in \mathcal{U}(M_n(\mathbb{C}))$ s.t.

$$T(x) = u^* x u, \quad x \in M_n(\mathbb{C}).$$

  (Kümmerer '83): Any unital qubit ($n = 2$) is a convex combination of automorphisms.

- Completely depolarizing channel S_n, $n \geq 2$

$$S_n(x) = \mathrm{tr}_n(x) 1_n, \quad x \in \mathbb{M}_n(\mathbb{C}).$$

- Schur multipliers associated to (complex) correlation matrices: If $B \in M_n(\mathbb{C})$ is a correlation matrix, then $T_B : M_n(\mathbb{C}) \to M_n(\mathbb{C})$

$$T_B\left([x_{ij}]_{1 \leq i,j \leq n}\right) = [x_{ij} b_{ij}]_{1 \leq i,j \leq n}, \quad [x_{ij}]_{1 \leq i,j \leq n} \in M_n(\mathbb{C}).$$

  is a unital quantum channel.

Definition (Anantharaman-Delaroche '05): A unital quantum channel $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *factorizable* if $\exists$ vN alg (N, ψ) with n.f. tracial state and unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$: $T = \beta^* \circ \alpha$.

$$\begin{tikzcd}
M_n(\mathbb{C}) \arrow[rr, "T"] \arrow[dr, "\alpha"'] & & M_n(\mathbb{C}) \arrow[dl, "\beta"'] \\
& M_n(\mathbb{C}) \otimes N \arrow[ur, bend right, "\beta^* = \beta^{-1}\circ\mathbb{E}_{\beta(M_n(\mathbb{C}))}"'] &
\end{tikzcd}$$

▶ α, β are injective (thus embeddings) and trace-preserving. Since unital embeddings of $M_n(\mathbb{C})$ into a vN alg are unitarily equiv, can take

$$\beta(x) = x \otimes 1_N, \quad \alpha(x) = u^*\beta(x)u, \quad x \in M_n(\mathbb{C}),$$

for some $u \in M_n(\mathbb{C}) \otimes N$ unitary. N can be taken II_1-vN alg (even factor).

Theorem (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel iff $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

Def/Thm (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel iff $\exists\ (N, \tau_N)$ finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

▶ (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

▶Automorphisms of $M_n(\mathbb{C})$ (unitarily implem channels) are factorizable.

Let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$, $n \geq 2$. Then $\mathcal{FM}(n)$ is convex and closed.

Further examples of factorizable channels:

- Convex comb of automorphisms of $M_n(\mathbb{C})$.
- The completely depolarizing channel S_n, as

$$\int_{\mathcal{U}(n)} u^* x u \, d\mu(u) = \mathrm{tr}_n(x) 1_n = S_n(x), \quad x \in M_n(\mathbb{C}).$$

- Schur multipliers associated to **real** correlation matrices (Ricard '08).

Theorem (Haagerup-M '11): For all $n \geq 3$, there exist non-factorizable quantum channels on $M_n(\mathbb{C})$. Each such channel violates the Asymptotic Quantum Birkhoff Conjecture of Smolin-Verstraete-Winter '05.

- Unital quantum channels which are extreme points of CPT or UCP, are non-factorizable. Concrete example: the Holevo-Werner channel W_3^-. With Haagerup and Ruskai, systematic recipe for non-factoriz channels.

- For a factorizable channel, "the" ancilla and its "size" **not** unique. E.g., possible ancillas for S_n are: $\mathbb{C}^{n^2}$, $M_n(\mathbb{C})$, but also (a corner of) $(M_n(\mathbb{C}), \mathrm{tr}_n) * (M_n(\mathbb{C}), \mathrm{tr}_n)$, the reduced free product von Neumann algebra of two copies of $M_n(\mathbb{C})$.

Question: Do we **need** (inf dim) vN alg to describe factorizable channels?

Let $\mathcal{FM}_{\text{fin}}(n) =$ factoriz channels on $M_n(\mathbb{C})$ admitting a finite dim ancilla.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

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Proposition (Haagerup-M '11): A Schur multiplier T_B is factorizable iff $B \in \mathcal{G}(n)$ (i.e., $B = [\tau(u_j^* u_i)]$, $u_1, \dots u_n$ unitaries in a fin vN alg (M, τ)). Furthermore,

$$T_B \in \mathcal{FM}_{\text{fin}}(n) \iff B \in \mathcal{G}_{\text{fin}}(n).$$

As the map $B \mapsto T_B$ is an affine homeo, the theorem above follows from non-closure of $\mathcal{G}_{\text{fin}}(n)$, whenever $n \geq 11$.

Thm (Haagerup-M '15) CEP pos **iff** $\overline{\mathcal{FM}_{\text{fin}}(n)} = \mathcal{FM}(n), \forall n \geq 3$.

(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C*-algebras):

- $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^* \, \iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the *canonical inclusions*. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some quantum channel T_τ. Furthermore, turns out that T_τ is factorizable!

In fact, the map $\Phi \colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

$$\Phi(T_{\rm fin}(M_n(\mathbb{C}) * M_n(\mathbb{C}))) = \mathcal{FM}_{\rm fin}(n),$$

where $T_{\rm fin}$ = tracial states that factor through fin. dim. C^*-alg.