The Connes Embedding Problem: from operator algebras to groups and quantum information theory

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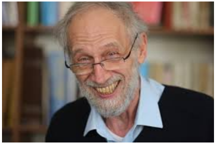


The Connes Embedding Problem (CEP)(Annals of Math.'76): Does every separable finite von Neumann alg M admit an embedding into

$$\mathcal{R}^\omega = \ell^\infty(\mathcal{R})/\{(T_n) : \lim_\omega \|T_n\|_2 = 0\},$$

$\omega =$ free ultrafilter on $\mathbb{N}$, $\|T\|_2 = \tau_{\mathcal{R}}(T^*T)^{1/2}$, $\tau_{\mathcal{R}} =$ trace on $\mathcal{R}$, the hyperfinite II_1-factor.

Theorem (Kirchberg '93): Let (M, τ) be a separable finite vN alg with faithful normal tracial state τ. Then M admits a τ-preserving embedding into $\mathcal{R}^\omega$ **iff** $\forall\ \varepsilon > 0$ and every set $u_1, \ldots, u_n$ of unitaries in M, $\exists\ k \geq 1$ and unitaries $v_1, \ldots, v_n$ in $M_k(\mathbb{C})$:

$$\left|\tau(u_j^* u_i) - \mathrm{tr}_k(v_j^* v_i)\right| < \varepsilon, \qquad 1 \leq i, j \leq n.$$

Consider the following sets of $n \times n$ matrices of correlations, $n \geq 2$:

$$
\begin{aligned}
\mathcal{G}_{\mathrm{matr}}(n) &= \bigcup_{k \geq 1} \left\{ \left[\mathrm{tr}_k(u_j^* u_i) \right] : u_1, \ldots, u_n \text{ unitaries in } M_k(\mathbb{C}) \right\}, \\
&\cap| \\
\mathcal{G}_{\mathrm{fin}}(n) &= \left\{ \left[\tau(u_j^* u_i) \right] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite dim C}^*\text{-alg } (\mathcal{A}, \tau) \right\}, \\
&\cap| \\
\mathcal{G}(n) &= \left\{ \left[\tau(u_j^* u_i) \right] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \right\}.
\end{aligned}
$$

All sets equal if $n = 2$.

Related: $D_{\mathrm{matr}}(n) \subseteq D_{\mathrm{fin}}(n) \subseteq D(n)$ where unitaries are replaced by proj.

Theorem (Kirchberg '93): CEP pos **iff** $\mathcal{G}(n) = \mathrm{cl}(\mathcal{G}_{\mathrm{matr}}(n))$, $\forall n \geq 3$.

Theorem (Rørdam-M '19):

1) $\mathcal{G}_{\mathrm{matr}}(n)$ is neither convex, nor closed when $n \geq 3$.
2) $\mathcal{G}_{\mathrm{fin}}(n)$ is convex for all $n \geq 2$, but not closed when $n \geq 11$.
3) $D_{\mathrm{fin}}(n)$ is convex for all $n \geq 2$, but not closed when $n \geq 5$.

A **trick** (originating in ideas of Regev-Slofstra-Vidick):

Let $p_1, \dots, p_n$ be projections in a vN alg (M, τ_M) with n.f. tracial state. Define unitaries $u_0, u_1, \dots u_{2n} \in M$ by $u_0 = 1$ and

$$u_j = 2p_j - 1,\ 1 \le j \le n, \quad u_j = \frac{1}{\sqrt{2}}(u_{j-n} + i \cdot 1),\ n+1 \le j \le 2n.$$

Let (N, τ_N) be some other vN alg with n.f. tracial state. Then $\exists$ unitaries $v_0, v_1, \dots, v_{2n} \in N$ s.t. $\tau_N(v_j^* v_i) = \tau_M(u_j^* u_i), \forall\, 0 \le i, j \le 2n$, iff $\exists$ projections $q_1, \dots, q_n \in N$ satisfying

$$\tau_N(q_j q_i) = \tau_M(p_j p_i), \qquad 1 \le i, j \le n.$$

- Recall: If $u \in A$ (unital C*-alg) unitary, then $\frac{1}{\sqrt{2}}(u + i \cdot 1)$ is a unitary **iff** u is a symmetry, i.e., $\frac{1}{2}(u+1)$ is a proj.

- Idea behind the **trick**: the map $u_j \mapsto v_j$, extended linearly between Eucl spaces $(\text{Span}\{u_0, \dots u_{2n}\}, \langle \cdot, \cdot \rangle_{\tau_M})$, $(\text{Span}\{v_0, \dots v_{2n}\}, \langle \cdot, \cdot \rangle_{\tau_N})$ is an isometry.

Let $p_1, \ldots, p_n$ be projections in a vN alg (M, τ_M) with n.f. tracial state. Define unitaries $u_0, u_1, \ldots u_{2n} \in M$ by $u_0 = 1$ and

$$u_j = 2p_j - 1,\ 1 \leq j \leq n, \quad u_j = \frac{1}{\sqrt{2}}(u_{j-n} + i \cdot 1),\ n+1 \leq j \leq 2n.$$

Let (N, τ_N) be some other vN alg with n.f. tracial state. Then $\exists$ unitaries $v_0, v_1, \ldots, v_{2n} \in N$ s.t. $\tau_N(v_j^* v_i) = \tau_M(u_j^* u_i), \forall\, 0 \leq i, j \leq 2n$, iff $\exists$ projections $q_1, \ldots, q_n \in N$ satisfying

$$\tau_N(q_j q_i) = \tau_M(p_j p_i), \qquad 1 \leq i, j \leq n.$$

Corollary: If $[\tau_M(p_j p_i)] \in \overline{\mathcal{D}_{\text{fin}}(n)} \setminus \mathcal{D}_{\text{fin}}(n)$, then the corresponding $2n+1$ unitaries satisfy $[\tau_M(u_j^* u_i)] \in \overline{\mathcal{G}_{\text{fin}}(2n+1)} \setminus \mathcal{G}_{\text{fin}}(2n+1)$.

- This proves "$D_{\mathrm{fin}}(n)$ not closed $\Rightarrow \mathcal{G}_{\mathrm{fin}}(2n+1)$ not closed".
- To prove $\mathcal{G}_{\mathrm{matr}}(n)$ not closed, $n \geq 3$, note that $D_{\mathrm{matr}}(n)$ not closed for $n \geq 1$, and use the **trick**.

To prove $D_{\rm fin}(n)$ not closed, $n \geq 5$, we followed Dykema-Paulsen-Prakash '17, and employed a theorem of Kruglyak-Rabanovich-Samoilenko '02, concerning existence of projections on a Hilbert space adding up to a scalar multiple of the identity, to show:

Theorem: Let $n \geq 5$ and $t \in [\frac{1}{2}(1-\sqrt{1-4/n}), \frac{1}{2}(1+\sqrt{1-4/n})]$. Define $A_t^{(n)} = [A_t^{(n)}(i,j)]_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ by

$$A_t^{(n)}(i,i) = t, \quad A_t^{(n)}(i,j) = \frac{t(nt-1)}{n-1}, i \neq j.$$

If $t \notin \mathbb{Q}$, then $A_t^{(n)} \in \overline{D_{\rm fin}(n)} \setminus D_{\rm fin}(n)$.

► (PSSTW '16): $\mathcal{D}(n)$, $\mathcal{D}_{\mathsf{fin}}(n)$ affinely homeo to the sets of *synchronous* quantum correlations $C_{qc}^s(n,2)$, $C_q^s(n,2)$. ($C_q^s(n)$ is rel. closed in C_q.)

Projections adding up to a scalar multiple of the identity operator:

Let Σ_n be the set of $\alpha \geq 0$ for which $\exists$ projections $p_1, \ldots, p_n$ on a Hilbert space H such that $\sum_{j=1}^n p_j = \alpha \cdot I_H$.

- It is known that $\Sigma_n \subset \mathbb{Q}$, when $n \leq 4$.

Theorem (Kruglyak-Rabanovich-Samoilenko '02): Let $n \geq 5$. There exist projections $p_1, \ldots, p_n$ on a *finite dimensional* Hilbert space H so that $\sum_{j=1}^n p_j = \alpha \cdot I_H$ if and only if $\alpha \in \Sigma_n \cap \mathbb{Q}$. Furthermore,

$$\left[\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})\right] \subseteq \Sigma_n.$$

Note: The "only if" part is easy (with Tr standard trace on $B(H)$):

$$\sum_{j=1}^n p_j = \alpha \cdot I_H \implies \alpha \cdot \dim(H) = \sum_{j=1}^n \mathrm{Tr}(p_j).$$

For $n \geq 2$ and $1/n \leq t \leq 1$, consider the following $n \times n$ matrix:

$$A_t^{(n)}(i,j) = \begin{cases} t, & i = j, \\ \dfrac{t(nt-1)}{n-1}, & i \neq j. \end{cases}$$

Proposition: Let $(\mathcal{A}, \tau)$ be a unital C^*-alg with faithful tracial state τ, and $p_1, \ldots, p_n \in \mathcal{A}$ be projections. Set $\alpha = nt$.

- If

$$\tau(p_j p_i) = A_t^{(n)}(i,j), \qquad 1 \leq i,j \leq n,$$

then $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$. Moreover, if $t \notin \mathbb{Q}$, then $\dim(\mathcal{A}) = \infty$. (Even stronger, $\mathcal{A}$ has no finite dimens repres.)

- Respectively, if $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$, then $\exists m \geq 1$ and projections $\widetilde{p}_1, \ldots, \widetilde{p}_n \in M_m(\mathcal{A})$ such that

$$(\tau \otimes \mathrm{tr_m})(\widetilde{p}_j \widetilde{p}_i) = A_t^{(n)}(i,j), \qquad 1 \leq i,j \leq n.$$

Recall

$$A_t^{(n)}(i,j) = \begin{cases} t, & i = j, \\ \dfrac{t(nt-1)}{n-1}, & i \neq j. \end{cases}$$

Combining previous proposition with the K-R-S theorem, we get

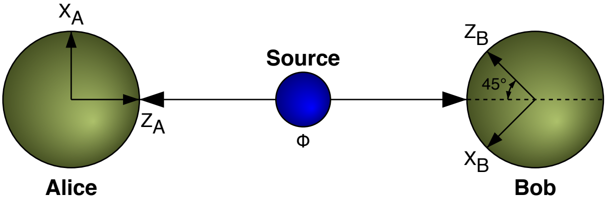
Theorem: Let $n \geq 5$, $t \in [\frac{1}{2}(1-\sqrt{1-4/n}), \frac{1}{2}(1+\sqrt{1-4/n})]$.

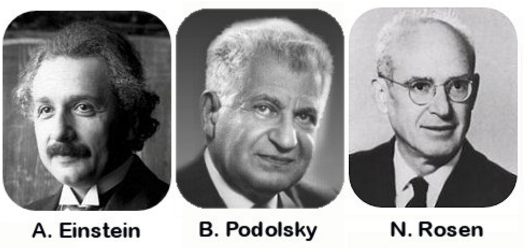
- If $t \in \mathbb{Q}$, then $A_t^{(n)} \in \mathcal{D}_{\mathrm{fin}}(n)$.
- If $t \notin \mathbb{Q}$, then $A_t^{(n)} \in \mathrm{cl}(\mathcal{D}_{\mathrm{fin}}(n)) \setminus \mathcal{D}_{\mathrm{fin}}(n)$.

In particular, $\mathcal{D}_{\mathrm{fin}}(n)$ is non-closed, when $n \geq 5$.

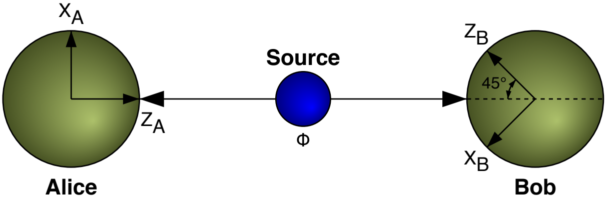
Note: If $t \in ((1-\sqrt{1-4/n})/2, (1+\sqrt{1-4/n})/2) \setminus \mathbb{Q}$, and $p_1, \ldots, p_n$ proj in a finite vN alg (N, τ_N) s.t. $\tau_N(p_j p_i) = A_t^{(n)}(i,j)$, $1 \leq i,j \leq n$, then N must be type II_1. **Ozawa**: Can take $N = \mathcal{R}$.

Quantum Correlations and The Einstein–Podolsky–Rosen paradox





Alice and Bob, residing in spatially separated labs, each receives a quantum system on which they can perform measurements.



Let's say that Alice and Bob can measure any one of n possible observables each with k possible outcomes. Let

$$P(a, b \mid x, y)$$

be the probability that Alice gets outcome a and Bob outcome b, when Alice measures observable x and Bob measures observable y.

Hidden variables - the classical model: $\exists$ prob. space (Ω, μ) and partitions $\{A_a^x\}_a$ and $\{B_b^y\}_b$ of Ω (one for each x, y) st

$$P(a, b \mid x, y) = \mu(A_a^x \cap B_b^y).$$

Hidden variables - the classical model: $\exists$ prob. space (Ω, μ) and partitions $\{A_a^x\}_a$ and $\{B_b^y\}_b$ of Ω (one for each x, y) s.t.

$$P(a, b \mid x, y) = \mu(A_a^x \cap B_b^y).$$

Definition: A PVM (projection valued measure) is a k-tuple $P_1, \ldots, P_k$ of projections on a Hilbert space H s.t. $\sum_{j=1}^k P_j = I$.

Two quantum models for interpreting the physical separation:

Tensor product: $\exists$ Hilbert spaces H_A, H_B, PVMs $\{P_a^x\}_a$, $\{Q_b^y\}_b$ on H_A, resp., H_B, and unit vector $\psi \in H_A \otimes H_B$ st

$$P(a, b \mid x, y) \;\; = \;\; \langle (P_a^x \otimes Q_b^y)\psi, \psi\rangle.$$

Commutativity: $\exists$ Hilbert space H, *commuting* PVMs $\{P_a^x\}_a$, $\{Q_b^y\}_b$ on H, and unit vector $\psi \in H$ st

$$P(a, b \mid x, y) \;\; = \;\; \langle P_a^x Q_b^y \psi, \psi\rangle.$$

Associated to these 3 models, we have the following convex sets of $nk \times nk$ matrices, rows are indexed by (a, x) and columns by (b, y):

$$\mathcal{C}_c(n,k) = \Big\{ \Big[\mu(A_a^x \cap B_b^y) \Big] : \{A_a^x\}_a, \{B_b^y\}_b \text{ partitions of } (\Omega, \mu) \Big\},$$

$$\mathcal{C}_{qs}(n,k) = \Big\{ \Big[\langle (P_a^x \otimes Q_b^y)\psi, \psi \rangle \Big] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVMs}, \psi \in H_A \otimes H_B \Big\},$$

$$\mathcal{C}_{qa}(n,k) = \mathrm{cl}(\mathcal{C}_{qs}(n,k)),$$

$$\mathcal{C}_{qc}(n,k) = \Big\{ \Big[\langle P_a^x Q_b^y \psi, \psi \rangle \Big] : \{P_a^x\}_a, \{Q_b^y\}_b \text{ PVMs}, [P_a^x, Q_b^y] = 0, \psi \in H \Big\}.$$

$\mathcal{C}_{qs}^{\mathrm{fin}}(n,k)$ and $\mathcal{C}_{qc}^{\mathrm{fin}}(n,k)$ denote the correlation sets, where the Hilbert spaces H_A, H_B, resp., H are *finite dimensional*.

$$\boxed{\begin{array}{ccccccccc} & & \mathcal{C}_{qs}^{\mathrm{fin}}(n,k) & & \overset{!}{=} & & \mathcal{C}_{qc}^{\mathrm{fin}}(n,k) & & \\ & & \cap\!| & & & & \cap\!| & & \\ \mathcal{C}_c(n,k) & \subseteq & \mathcal{C}_{qs}(n,k) & \subseteq & \mathcal{C}_{qa}(n,k) & \subseteq & \mathcal{C}_{qc}(n,k) & \subseteq & M_{nk}([0,1]) \end{array}}$$

- $\mathrm{cl}(\mathcal{C}_{qs}^{\mathrm{fin}}(n,k)) = \mathrm{cl}(\mathcal{C}_{qs}(n,k)) = \mathcal{C}_{qa}(n,k)$.

$$
\begin{array}{ccccccccc}
 & & \mathcal{C}_{qs}^{\mathrm{fin}}(n,k) & & \overset{!}{=} & & \mathcal{C}_{qc}^{\mathrm{fin}}(n,k) & & \\
 & & \cap\!| & & & & \cap\!| & & \\
\mathcal{C}_c(n,k) & \subseteq & \mathcal{C}_{qs}(n,k) & \subseteq & \mathcal{C}_{qa}(n,k) & \subseteq & \mathcal{C}_{qc}(n,k) & \subseteq & M_{nk}([0,1])
\end{array}
$$

- $\mathrm{cl}(\mathcal{C}_{qs}^{\mathrm{fin}}(n,k)) = \mathrm{cl}(\mathcal{C}_{qs}(n,k)) = \mathcal{C}_{qa}(n,k)$.

EPR–Bell's inequality–Aspect: $\mathcal{C}_c(n,k) \neq \mathcal{C}_{qs}(n,k)$. (This also follows from Grothendieck's ineq in Functional Analysis.)

Conjecture/Problem (Tsirelson): $\mathcal{C}_{qa}(n,k) \overset{?}{=} \mathcal{C}_{qc}(n,k)$. Equivalently,

$$\mathrm{cl}(\mathcal{C}_{qc}^{\mathrm{fin}}(n,k)) \overset{?}{=} \mathcal{C}_{qc}(n,k).$$

(Slofstra '16): $\mathcal{C}_{qs}(n,k) \neq \mathcal{C}_{qc}(n,k)$. He further showed ('17) that $\mathcal{C}_{qs}(n,k)$ is **not** closed, for n and k large enough, so $\mathcal{C}_{qs}(n,k) \neq \mathcal{C}_{qa}(n,k)$.

(Dykema-Paulsen-Prakash '17), (Rørdam-M '19): $\mathcal{C}_{qs}(5,2)$ **not** closed. [Proof by D-P-P uses nonlocal quantum games.]

Some background on C^*-tensor products and $C^*(\mathbb{F}_\infty)$:

$\mathbb{F}_\infty$ = free group with countably infinitely many generators.

$C^*(\mathbb{F}_\infty)$ = universal C^*-alg. generated by a sequence of unitaries.

- Every unital separable C^*-alg is a quotient of $C^*(\mathbb{F}_\infty)$.
- For unital C^*-algebras $A \subseteq B(H)$ and $B \subseteq B(K)$:

• $A \otimes_{\min} B \subseteq B(H \otimes K)$ = the *spatial tensor product* = the closure of the *algebraic* tensor product $A \odot B \subseteq B(H \otimes K)$

• $A \otimes_{\max} B$ = universal C^*-algebra generated by *commuting* copies of A and B

- In general we have canonical surjection: $A \otimes_{\max} B \to A \otimes_{\min} B$.
- $A \otimes_{\max} B = A \otimes_{\min} B$ if A or B is *nuclear*, but not in general.

- $\Gamma = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$ (n free factors).

Theorem (Fritz, Junge et. al. '09):

- $\mathcal{C}_{qa}(n,k) = \Big\{ \Big[\varphi(e_a^x \otimes e_b^y) \Big] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \Big\}$.
- $\mathcal{C}_{qc}(n,k) = \Big\{ \Big[\varphi(e_a^x \otimes e_b^y) \Big] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \Big\}$.

▶ $C^*(\Gamma) = C^*(\mathbb{Z}_k) *_1 C^*(\mathbb{Z}_k) *_1 \cdots *_1 C^*(\mathbb{Z}_k)$.

▶ $C^*(\mathbb{Z}_k) = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k$, where e_j are proj'n and $\sum_j e_j = 1$.

- Let $e_a^x \in C^*(\Gamma)$ be the projection e_a in the xth free factor above.

▶ If $\{P_a^x\}_a \subseteq B(H)$ are PVM's, then $\exists$ *-hom

$$\Phi\colon C^*(\Gamma) \to B(H) \text{ st } \Phi(e_a^x) = P_a^x \text{ for all } a, x.$$

▶ If $\{P_a^x\}_a, \{Q_b^y\}_b \subseteq B(H)$ are commuting PVM's, then $\exists$ *-hom

$$\Psi\colon C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \to B(H) \text{ st } \Psi(e_a^x \otimes e_b^y) = P_a^x Q_b^y \text{ for all } a, x, b, y.$$

- Let $\Gamma = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$ (n free factors), $n, k \geq 2$.

Theorem (Fritz/Junge et. al. '09):

- $\mathcal{C}_{qa}(n,k) = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \right\}$.
- $\mathcal{C}_{qc}(n,k) = \left\{ \left[\varphi(e_a^x \otimes e_b^y) \right] : \varphi \text{ state on } C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \right\}$.

- $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ is RFD [$\Rightarrow \mathcal{C}_{qs}^{\mathrm{fin}}(n,k) \overset{\text{dense}}{\subseteq} \mathcal{C}_{qs}(n,k)$].

Theorem (Kirchberg '93, Fritz/Junge et. al. '09, Ozawa '12): TFAE:

(i) $C^*(\Gamma) \otimes_{\max} C^*(\Gamma) = C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ for all $n, k \geq 2$,

(ii) $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$,

(iii) Connes embedding problem has positive answer,

(iv) Tsirelson's conjecture is true.