# The Connes Embedding Problem: from operator algebras to groups and quantum information theory

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The theory of von Neumann algebras, introduced by J. von Neumann in 1929-1930 as *rings of operators*, was developed with F. Murray in a series of papers 1936-1943 as mathematical framework for quantum mechanics, where Heisenberg's uncertainty relation is expressed as noncommutativity of certain operators.

**Definition:** A von Neumann algebra $M$ is subalgebra of $\mathcal{B}(H)$, the set of all bounded linear operators on a Hilbert space $H$, containing the unit, closed under taking adjoints: $T \in M \implies T^* \in M$, and closed in the strong operator topology (SOT): $T_n \to T$ iff $T_n\psi \to T\psi$, $\psi \in H$.

von Neumann's bicommutant theorem: $M \subseteq \mathcal{B}(H)$ is a vN alg **iff** $M = M^*$ and $M'' = M$, where $M' = \{S \in \mathcal{B}(H) : TS = ST \text{ for all } T \in M\}$.

von Neumann algebras $\sim$ *non-commutative measure spaces*

**Definition:** A sep vN alg $M$ is finite if it has a faithful *tracial state*: pos lin functional $\tau\colon M \to \mathbb{C}$ so that $\tau(ST) = \tau(TS)$, $S, T \in M$, $\tau(I) = 1$.

- $M$ is type $\mathrm{II}_1$ if it is finite and has no finite dimens representations.
- $M$ is a factor if it has trivial center.

**Examples**: Let $\Gamma$ countable infinite group. Consider its left-regular representation $\lambda\colon \Gamma \to \mathcal{U}(\ell^2(\Gamma))$, $\lambda(g)\delta_x = \delta_{gx}$, $g, x \in \Gamma$, where $\{\delta_g\}_{g\in\Gamma}$ ONB in $\ell^2(\Gamma)$. Set

$$\mathcal{L}(\Gamma) \;=\; \overline{\mathrm{span}(\lambda(\Gamma))}^{\mathrm{SOT}} \subseteq \mathcal{B}(\ell^2(\Gamma)).$$

- $\mathcal{L}(\Gamma)$ finite vN alg with faithf. tracial state $\tau(T) = \langle T\delta_e, \delta_e\rangle$, $T \in \mathcal{L}(\Gamma)$. It is a $\mathrm{II}_1$-factor iff $\Gamma$ is icc (infinite conjugacy classes).

**Definition:** A vN alg $M$ is *hyperfinite* if $\exists F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq M$ s.t. each $F_j$ is *finite dimensional* and $\bigcup_{j=1}^{\infty} F_j$ is SOT- dense in $M$.

- (Murray-von Neumann '40): There is **unique** *hyperfinite* type $\mathrm{II}_1$ factor, denoted $\mathcal{R}$. (Can realize $\mathcal{R}$ as $\mathcal{L}(S_\infty)$, $S_\infty$= finitely supported perm on $\mathbb{N}$.)

**Constructing finite von Neumann algebras from tracial $C^*$-algebras**:

Let $A$ be a unital $C^*$-alg with a tracial state $\tau$, and let $\pi_\tau \colon A \to B(H)$ be the GNS repres s.t. $\tau(a) = \langle \pi_\tau(a)\xi, \xi\rangle$, for some cyclic unit vector $\xi \in H$.

Then $\tau$ extends to a *normal* faithful tracial state on $\pi_\tau(A)'' \subseteq B(H)$, given by $\overline{\tau}(x) = \langle x\xi, \xi\rangle$, $x \in \pi_\tau(A)''$, so $\pi_\tau(A)''$ is a finite von Neumann alg.

• $\pi_\tau(A)''$ is a factor iff $\tau$ is an extreme point in $T(A)$ (e.g., if $\tau$ is the unique trace on $A$); it's a $\mathrm{II}_1$-factor if, moreover, $\pi_\tau(A)''$ inf. dim.

• $\pi_\tau(A)''$ is the hyperfinite $\mathrm{II}_1$-factor $\mathcal{R}$ if $A$ is (any) UHF-algebra with $\tau =$ the unique trace on $A$. E.g., $A = \varinjlim \left(M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \cdots\right)$

**Proof:** $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ (norm-closure) with $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ finite dim subalgebras, so $\pi_\tau(A)'' = \overline{\bigcup_{n=1}^{\infty} \pi(A_n)}$ (SOT-closure).

► More generally (and much deeper!): $\pi_\tau(A)''$ is hyperfinite whenever $A$ is nuclear, hence $\pi_\tau(A)'' = \mathcal{R}$ if also $\tau \in \partial_e T(A)$ and $\pi_\tau(A)''$ inf. dim.

## Ultrapowers of finite von Neumann algebras

Let $(M, \tau)$ a vN alg with n.f.t.s. $\tau$, and let $\omega =$ free ultrafilter on $\mathbb{N}$. Set

$$I^\omega = \left\{ \{x_n\}_{n\geq 1} \in \ell^\infty(M) : \lim_\omega \|x_n\|_{2,\tau} = 0 \right\} \lhd \ell^\infty(M).$$

Set $M^\omega = \ell^\infty(M)/I^\omega$, and let $\tau_\omega$ be the tracial state on $M^\omega$ given by

$$\tau_\omega(\pi_\omega(\{x_n\}_{n\geq 1})) = \lim_\omega \tau(x_n), \qquad \{x_n\}_{n\geq 1} \in \ell^\infty(M),$$

where $\pi_\omega \colon \ell^\infty(M) \to M^\omega$ is the quotient mapping.

**Proposition:** $M^\omega$ is a von Neumann algebra, and $\tau_\omega$ is a n.f.t.s. on $M^\omega$. If $M$ is a $\mathrm{II}_1$-factor, then so is $M^\omega$.

This non-trivial fact follows from the two results below:

**Theorem:** Let $(M, \tau)$ be a vN alg with n.f.t.s. $\tau$. Then the unit ball of $M$ and $\mathcal{U}(M)$ are both complete wrt $\|\cdot\|_{2,\tau}$, where $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$.

Conversely, if $A$ is a unital $C^*$-alg with faithful tracial state $\tau$ s.t. the unit ball in $A$ is complete wrt $\|\cdot\|_{2,\tau}$, then $A$ is a vN alg and $\tau$ is normal.

**Lemma:** The unit ball of $M^\omega$ is complete wrt $\|\cdot\|_{2,\tau_\omega}$.

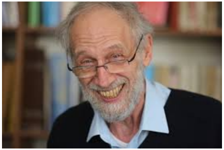
▶ One can in a similar way, for any sequence $\{k_n\}_{n\geq 1}$ of pos. integers, define the ultraproduct $\prod^\omega M_{k_n}(\mathbb{C})$ of the seq $(M_{k_n}(\mathbb{C}), \mathrm{tr}_{k_n})$ by

$$\prod^{\omega} M_{k_n}(\mathbb{C}) := \prod_{n=1}^{\infty} M_{k_n}/I^\omega, \quad I^\omega = \Big\{ \{a_n\}_{n\geq 1} \in \prod_{n=1}^{\infty} M_{k_n} : \lim_{\omega} \|a_n\|_2 = 0 \Big\},$$

which again is a $\mathrm{II}_1$-factor (if $k_n \to \infty$).

▶ The theorem on the previous slide also implies the following useful fact:

**Bonus proposition:** Let $(M, \tau_M)$ and $(N, \tau_N)$ be two vN algs with n.f.t.s. $\tau_M$ and $\tau_N$, resp. Then any unital trace-preserving $^*$-hom $\varphi\colon M \to N$ is automatically normal and $\varphi(M)$ is a von Neumann algebra.



**The Connes Embedding Problem (CEP)**(Annals of Math, 1976): Does every separable finite vN alg $M$ admit an embedding into

$$\mathcal{R}^\omega = \ell^\infty(\mathcal{R})/\{(T_n) : \lim_\omega \|T_n\|_2 = 0\},$$

$\omega =$ free ultrafilter on $\mathbb{N}$, $\|T\|_2 = \tau_\mathcal{R}(T^*T)^{1/2}$, $\tau_\mathcal{R} =$ trace on $\mathcal{R}$.

**CEP** (Reformulation): Does every separable finite vN alg $(M, \tau)$ admit an "approximate embedding" into a matrix algebra: $\forall N, k \geq 1 \, \forall \varepsilon > 0$, $\forall$ unitaries $u_1, \ldots, u_k \in M$, $\exists n \geq 1$ $\exists$ unitaries $v_1, \ldots, v_k \in M_n(\mathbb{C})$ s.t.

$$\left| \tau\left(u_{i_1}^{\nu_1} u_{i_2}^{\nu_2} \cdots u_{i_r}^{\nu_r}\right) - \mathrm{tr}_\mathrm{n}\left(v_{i_1}^{\nu_1} v_{i_2}^{\nu_2} \cdots v_{i_r}^{\nu_r}\right) \right| < \varepsilon$$

$\forall r \geq 1 \, \forall i_1, \ldots, i_r \in \{1, \ldots, k\} \, \forall \nu_1, \ldots, \nu_r \in \mathbb{Z}$ with $\sum |\nu_j| \leq N$.

**Theorem:** Let $(M, \tau)$ be a separable finite vN alg with n.f.t.s. $\tau$, and let $\omega$ be a free ultrafilter on $\mathbb{N}$. Then $\exists$ a trace-preserving $^*$-hom $M \to \mathcal{R}^\omega$ (necessarily normal) **iff** $\exists\, k_n \geq 1$, $\exists$ maps $\varphi_n \colon M \to M_{k_n}(\mathbb{C})$, $n \geq 1$, s.t.

(i) $\varphi_n(1_M) = 1_{k_n}$,

(ii) $\lim_{n\to\omega} \|\varphi_n(\alpha x + y) - \alpha\varphi_n(x) - \varphi_n(y)\|_2 = 0$, for $x, y \in M$, $\alpha \in \mathbb{C}$,

(iii) $\lim_{n\to\omega} \|\varphi_n(xy) - \varphi_n(x)\varphi_n(y)\|_2 = 0$, for $x, y \in M$,

(iv) $\lim_{n\to\omega} \|\varphi_n(x^*) - \varphi_n(x)^*\|_2 = 0$, for $x \in M$,

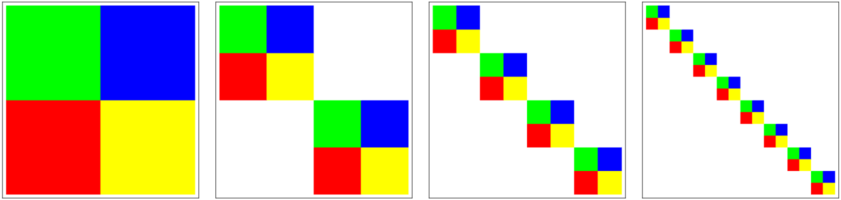
(v) $\lim_{n\to\omega} \mathrm{tr}_{k_n}(\varphi_n(x)) = \tau(x)$, for $x \in M$,

(vi) $\sup_n \|\varphi_n(x)\| < \infty$, for $x \in M$,

where $\|a\|_2 = \mathrm{tr}_{k_n}(a^*a)^{1/2}$, for $a \in M_{k_n}(\mathbb{C})$.

**The Connes Embedding Problem:**

# $\mathcal{R}$ We Living in the Matrix?



*Roy Araiza and Rolando de Santiago*

*Notices AMS*, Volume 66, No. 8, 2016.

Two approximation properties for countable discrete groups $\Gamma$:

$\Gamma$ is sofic (after Gromov) if it "admits an approximate embedding into the symmetric groups $S_n$", i.e., if $\forall F \Subset \Gamma\ \forall \varepsilon > 0\ \exists n \geq 1\ \exists \varphi \colon \Gamma \to S_n$ s.t.

- $d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon$, for all $g, h \in F$,
- $d_F(\varphi(g), \varphi(h)) \geq 1 - \varepsilon$, for all $g \neq h \in F$.

$d_F$ is the *Hamming metric*:

$$d_F(\alpha, \beta) = |\{j \in F : \alpha(j) \neq \beta(j)\}|/|F|, \quad \alpha, \beta \in \mathrm{Sym}(F).$$

$\Gamma$ is Connes-embeddable if it "admits an approximate embedding into the unitary groups $U(n)$ of $M_n(\mathbb{C})$", i.e., if $\forall F \Subset \Gamma\ \forall \varepsilon > 0\ \exists n \geq 1$ $\exists \varphi \colon \Gamma \to U(n)$ s.t.

- $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$, for all $g, h \in F$,
- $\|\varphi(g) - \varphi(h)\|_2 \geq \sqrt{2} - \varepsilon$, for all $g \neq h \in F$.

- Not known if all groups are Connes-embeddable (or sofic).

$$
\begin{tikzcd}
\text{Linear \& fin gen} \arrow[r, Rightarrow] & \text{RF} \arrow[r, Rightarrow] & \text{LEF} \arrow[d, Rightarrow] & \\
& \text{Amenable} \arrow[r, Rightarrow] & \text{Sofic} \arrow[r, Rightarrow] & \text{Connes-embeddable}
\end{tikzcd}
$$

RF= Residually finite (= separating family of homs. into finite groups)

LEF = Locally Embeddable into Finite groups.

**Theorem** (Radulescu): $\Gamma$ is Connes-embeddable **iff** $\mathcal{L}(\Gamma)$ embeds into $\mathcal{R}^\omega$.

- **Affirmative** answer to CEP implies that **all** groups are Connes-embedd (but not necessarily sofic), while a **negative** answer does **not** imply the existence of a non-Connes-embeddable group, nor of a non-sofic one.

**Theorem** (Kirchberg '93): Let $(M, \tau)$ separable finite von Neumann alg with normal faithful tracial state $\tau$. Then $M$ admits a trace-preserving embedding into $\mathcal{R}^\omega$ **iff** $\forall\, \varepsilon > 0$ and every set $u_1, \ldots, u_n$ of unitaries in $M$, $\exists\ k \geq 1$ and unitaries $v_1, \ldots, v_n$ in $M_k(\mathbb{C})$ :

$$\left|\tau(u_j^* u_i) - \mathrm{tr}_k(v_j^* v_i)\right| < \varepsilon, \qquad 1 \leq i, j \leq n.$$

(Dykema-Juschenko): Cons. sets of $n \times n$ matrices of correlations, $n \geq 2$:

$$\begin{array}{rcl} \mathcal{G}_{\mathrm{matr}}(n) & = & \displaystyle\bigcup_{k\geq 1} \left\{ \left[\mathrm{tr}_k(u_j^* u_i)\right] : u_1, \ldots, u_n \text{ unitaries in } M_k(\mathbb{C}) \right\}, \\ \cap\!\mid & & \\ \mathcal{G}(n) & = & \left\{ \left[\tau(u_j^* u_i)\right] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \right\}. \end{array}$$

**Theorem** (Kirchberg '93): CEP pos **iff** $\mathcal{G}(n) = \mathrm{cl}(\mathcal{G}_{\mathrm{matr}}(n))$, $\forall n \geq 3$.

- It's non-trivial that $\mathcal{G}_{\mathrm{matr}}(n)$ is not closed, when $n \geq 3$ (Rørdam-M '19).

**Proof** of $\Leftarrow$ in Kirchberg's thm: uses Jordan homs between $C^*$-algebras.

**Definition**: Let $A, B$ be C*-algs. A linear map $\varphi\colon A \to B$ is a Jordan *-homomorphism if it is self-adjoint and $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$, $a, b \in A$, where $a \circ b := \frac{1}{2}(ab + ba)$.

**Remarks/Definitions**: If $a, b \in A$ are self-adj, then so is $a \circ b$. Restricting a Jordan *-hom $\varphi\colon A \to B$ to $A_{\mathrm{sa}}$ gives an $\mathbb{R}$-linear map $\varphi'\colon A_{\mathrm{sa}} \to B_{\mathrm{sa}}$, which preserves the Jordan product. We call such map a Jordan hom.

Conversely, any Jordan hom $\varphi'\colon A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ can uniquely be extended to a $\mathbb{C}$-linear map $\varphi\colon A \to B$, and (can check) $\varphi$ is a Jordan *-hom.

An anti-*-homomorphism $\varphi\colon A \to B$ is a linear self-adj map satisfying $\varphi(ab) = \varphi(b)\varphi(a)$, for all $a, b \in A$ (i.e., an *anti-*-hom* $A \to B$ is a *-hom $A \to B^{\mathrm{op}}$).

Any *-hom and any anti-*-hom is a Jordan *-hom. Størmer proved that any Jordan *-hom between unital $C^*$-algs is a sum of a *-hom and an anti-*-hom.

**Theorem** (Størmer): Let $A, B$ be unital C$^*$-algs with $B \subseteq \mathcal{B}(H)$, and $\varphi\colon A \to B$ a unital Jordan $^*$-hom. Then $\exists\, p \in B'' \cap \varphi(A)'$ projection s.t. $a \in A \mapsto \varphi(a)p \in \mathcal{B}(H)$ is a $^*$-hom and $a \in A \mapsto \varphi(a)(I_H - p) \in \mathcal{B}(H)$ is an anti-$^*$-hom.

**Definition**: Let $A, B$ unital C$^*$-algs, $\varphi\colon A \to B$ unital pos contraction. Set

$$\text{J-Mult}(\varphi) = \{a \in A_{\mathrm{sa}} : \varphi(a^2) = \varphi(a)^2\}.$$

**Proposition**: Let $A, B$ unital C$^*$-algs, $\varphi\colon A \to B$ unital pos contraction. If $a \in \text{J-Mult}(\varphi)$, then $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$, for all $b \in A_{\mathrm{sa}}$. Furthermore, J-Mult$(\varphi)$ is a Jordan subalgebra of $A_{\mathrm{sa}}$ (= closed $\mathbb{R}$-linear subset of $A_{\mathrm{sa}}$, closed under the Jordan product $\circ$).

**Cor**: A self-adj lin $\varphi\colon A \to B$ is Jordan $^*$-hom **iff** $\varphi(a^2) = \varphi(a)^2, \forall a \in A_{\mathrm{sa}}$.

**Lemma**: Let $A, B$ unital C$^*$-algs, $\varphi\colon A \to B$ unital pos contr. Assume that the linear span of projections is dense in $A$ (e.g., if $A$ is a vN alg). TFAE:

(1) $\varphi$ is a Jordan $^*$-hom.
(2) $\varphi$ maps unitaries in $A$ to unitaries in $B$.
(3) $\varphi$ maps projections in $A$ to projections in $B$.

**Proof** of Kirchberg's theorem:

$\Leftarrow$: May assume that the unitaries $v_1, \ldots, v_n$ belong to $\mathcal{R}$ (as there exist unital trace-preserv embeddings $M_k(\mathbb{C}) \to \mathcal{R}$, $k \geq 1$).

Let $u_1 := 1, u_2, u_3, \ldots$ $\|\cdot\|_2$-dense sequence of unitaries in $M$ (by sep).

By hypothesis, $\forall\ n \geq 1$, $\exists$ unitaries $v_1^{(n)}, \ldots, v_n^{(n)} \in \mathcal{R}$ with $v_1^{(n)} = 1$ and

$$\left|\tau(u_j^* u_i) - \tau_{\mathcal{R}}((v_j^{(n)})^* v_i^{(n)})\right| < 1/n, \qquad 1 \leq i, j \leq n.$$

Set $v_j^{(n)} = 1$, when $j > n$, and $v_j = \pi_\omega(\{v_j^{(n)}\}_{n\geq 1}) \in \mathcal{R}^\omega$, else. Then $v_1 = 1, v_2, v_3, \ldots$ are unitaries in $\mathcal{R}^\omega$ satisfying

$$\tau_{\mathcal{R}^\omega}(v_j^* v_i) = \lim_\omega \tau_{\mathcal{R}}\big((v_j^{(n)})^* v_i^{(n)}\big) = \tau(u_j^* u_i), \qquad i, j \geq 1. \qquad (*)$$

Since $u_1 = 1 = v_1$, deduce $\tau(u_i) = \tau_{\mathcal{R}^\omega}(v_i)$, $i \geq 1$, by taking $j = 1$ in $(*)$.

View $M$, $\mathcal{R}^\omega$ as Euclidean spaces wrt inner product given by $\tau$, resp, $\tau_{\mathcal{R}^\omega}$. Then $(*)$ becomes $\langle v_i, v_j\rangle_{\mathcal{R}^\omega} = \langle u_i, u_j\rangle_{\tau_{\mathcal{R}}}$, $i, j \geq 1$. For each fin supported seq $(\alpha_j)_{j\geq 1}$ in $\mathbb{C}$, we get $\|\sum_{j=1}^\infty \alpha_j u_j\|_2 = \|\sum_{j=1}^\infty \alpha_j v_j\|_2$. Thus $\exists$ unique $\|\cdot\|_2$-isometric map $\varphi_0\colon \mathrm{span}\{u_1, u_2, \ldots\} \to \mathrm{span}\{v_1, v_2, \ldots\}$, satisfying

$$\varphi_0(u_j) = v_j, \quad j \geq 1.$$

Since $\forall r > 0$, the closed balls $(M)_r$, $(\mathcal{R}^\omega)_r$ of $M$, $\mathcal{R}^\omega$ are $\|\cdot\|_2$-complete, $\varphi_0$ extends to an $\|\cdot\|_2$-isometric linear map $\varphi\colon M \to \mathcal{R}^\omega$.

Note $\varphi(1_M) = \varphi(u_1) = v_1 = 1_{\mathcal{R}^\omega}$. As $\mathcal{U}(M)$, $\mathcal{U}(\mathcal{R}^\omega)$ are closed in $\|\cdot\|_2$, it follows that $\varphi$ maps unitaries in $M$ to unitaries in $\mathcal{R}^\omega$. An application of the Russo-Dye theorem (the closed convex hull of unitaries in any unital $C^*$-alg is dense in its closed unit ball) gives

$$\|\varphi\| = \sup\{\|\varphi(u)\| : u \in \mathcal{U}(M)\} = 1 = \|\varphi(1_M)\|.$$

This shows $\varphi$ unital positive contraction, hence Jordan $^*$-hom, by Lemma. Moreover, $\tau_{\mathcal{R}^\omega}(\varphi(x)) = \tau_M(x)$, whenever $x \in \{u_1, u_2, \dots\}$. By continuity of traces, this holds $\forall\ x \in \mathcal{U}(M)$, hence $\forall\ x \in M$, so $\varphi$ is trace-preserving.

By Størmer's theorem, $\exists$ projection $p \in \mathcal{R}^\omega \cap \varphi(M)'$ s.t. if $\varphi_1(x) = \varphi(x)p$ and $\varphi_2(x) = \varphi(x)(1-p)$, for $x \in M$, then $\varphi_1\colon M \to p\mathcal{R}^\omega p$ is a unital $^*$-hom, while $\varphi_2\colon M \to (1-p)\mathcal{R}^\omega(1-p)$ is a unital anti-$^*$-hom.

It is (well)-known that $\mathcal{R}$ is isomorphic to its opposite vN alg $\mathcal{R}^{\mathrm{op}}$ (holds for any group vN alg). An isomorphism $\mathcal{R} \to \mathcal{R}^{\mathrm{op}}$ induces naturally an isomorph $\rho\colon \mathcal{R}^\omega \to \left(\mathcal{R}^\omega\right)^{\mathrm{op}}$. Then the map $x \in M \mapsto \varphi_1(x) + (\rho \circ \varphi_2)(x)$ defines a unital trace-preserving $^*$-hom $M \to \mathcal{R}^\omega$, as desired. $\square$

To prove the reverse implication $\Leftarrow$, it suffices to show:

$\forall\, u_1, \ldots, u_n \in \mathcal{U}(\mathcal{R}^\omega)\ \forall\, \varepsilon > 0$, $\exists k \geq 1\ \exists v_1, \ldots, v_n \in \mathcal{U}(M_k(\mathbb{C}))$ s.t.

$$|\tau(u_j^* u_i) - \mathrm{tr}_k(v_j^* v_i)| < \varepsilon, \quad \forall 1 \leq i, j \leq n.$$

**Step 1:** Unitaries lift (to unitaries) from any quotient of a finite vN alg, so $\exists w_1, \ldots, w_n \in \mathcal{U}(\ell^\infty(\mathcal{R}))\colon \pi(w_j) = u_j$, $\pi\colon \ell^\infty(\mathcal{R}) \to \mathcal{R}^\omega$ quotient map.

**Step 2:** Write $w_j = \{w_j(m)\}_{m \geq 1}$ with $w_j(m) \in \mathcal{U}(\mathcal{R})$. Note that

$$\tau(u_j^* u_i) = \lim_{m \to \omega} \tau_{\mathcal{R}}(w_j(m)^* w_i(m)).$$

Hence $\exists m \geq 1$ s.t. $|\tau(u_j^* u_i) - \tau_{\mathcal{R}}(w_j(m)^* w_i(m))| < \varepsilon/2$, for $1 \leq i, j \leq n$.

**Step 3:** $\mathcal{R}$ hyperfinite, so $\exists A_1 \subseteq A_2 \subseteq \cdots \subseteq \mathcal{R}$ s.t. $A_r \cong M_{k_r}(\mathbb{C})$ and $\bigcup_{r \geq 1} A_r$ is SOT-dense in $\mathcal{R}$.

**Step 4:** By Kaplanski's density thm, $\exists r \geq 1$ and $v_1, \ldots, v_n \in \mathcal{U}(A_r)$ s.t. $\|w_j(m) - v_j\|_2 < \varepsilon/4$, so $|\tau_{\mathcal{R}}(w_j(m)^* w_i(m)) - \tau_{\mathcal{R}}(v_j^* v_i)| \leq \varepsilon/2$.

**Step 5:** Set $k = k_r$ and identify $(A_r, \tau_{\mathcal{R}})$ with $(M_k(\mathbb{C}), \mathrm{tr}_k)$. $\square$