
# On the stochastic operators on $L^1$

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Mount Allison University

June 18, 2021

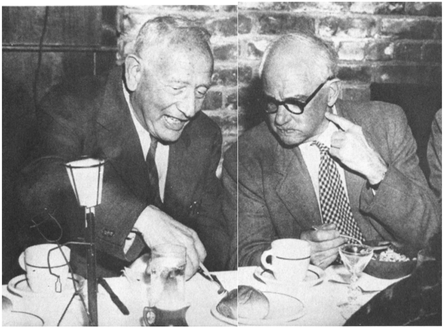
# Presentation Outline

# Short History

- **1929: Hardy**

- **1929: Hardy, Littlewood**

- **1929:** **Hardy, Littlewood and Pólya**



# Definition of Vector Majorization

Let $X = (x_1, x_2, \ldots, x_n)$ be a real vector. $X$ has been reordered so that $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$.

**Definition 1 ( 1929- Hardy, Littlewood and Pólya [4] )**

if $X, Y \in \mathbb{R}^n$, we say $X$ **is *majorized* by** $Y$, dtenoted $X \prec Y$, if

$$\begin{array}{rcll} x_1^{\downarrow} & \leq & y_1^{\downarrow} & \\ x_1^{\downarrow} + x_2^{\downarrow} & \leq & y_1^{\downarrow} + y_2^{\downarrow} & \\ & \vdots & & \\ \sum_{j=1}^{k} x_j^{\downarrow} & \leq & \sum_{j=1}^{k} y_j^{\downarrow}, & k \in \{1, \ldots, n-1\} \\ \textit{and} \quad \sum_{j=1}^{n} x_j & = & \sum_{j=1}^{n} y_j. & \end{array}$$

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# Equivalent Conditions for Vector Majorization

## Theorem 2 ( 1934- Hardy, Littlewood, and Pólya [4])

*For $X, Y \in \mathbb{R}^n$ the followings are equivalent.*

(1) $X \prec Y$,

(2) *There exists a doubly stochastic matrix $D_{n\times n}$ such that* $X = DY$.

(3) $\sum_{i=1}^{n} \varphi(x_i) \leq \sum_{i=1}^{n} \varphi(y_i)$, *holds for all continuous convex functions* $\varphi : \mathbb{R} \to \mathbb{R}$.

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# Presentation Outline

In 1999, Nielsen used vector majorization to link problem of state transformation with mathematics in a finite dimensional system.

Any isolated physical system is identified with some finite or infinite dimensional Hilbert spaces and its pure states, which system can be described completely by one of them, correspond to unit vectors (for details see [8, section 2.2.1]).

The state space of a composite system is modelled by the tensor product of subsystems (see [8, section 2.2.8]).

We will denote the unit column vector $\phi$ in Hilbert space $H$, with "Ket" $|\phi\rangle$.

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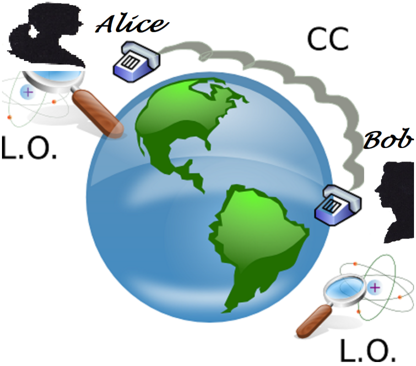
# Local Operations and Classical Communication

The parties are not allowed to exchange particles coherently.
Only local operations and classical communication is allowed.



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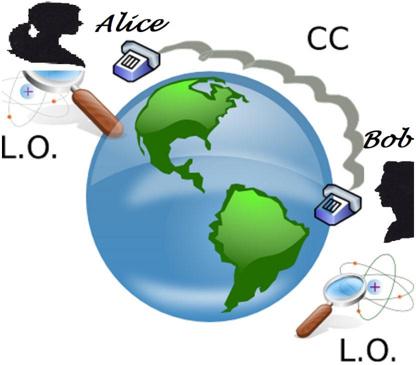

# Nielsen's Theorem in the finite dimensional

## Theorem 3 (Nielsen's Theorem [8])

*$|\psi\rangle$ can be converted to $|\phi\rangle$ by LOCC channel if and only if $\lambda_\psi \prec \lambda_\phi$.*

## Theorem 4 (Schmidt decomposition; infinite case)

*For every $|\psi\rangle \in H_a \otimes H_b$ there exist orthonormal Schmidt sets (not necessarily basis) $\{|e_i\rangle\}_{i=1}^{\infty} \subset H_a$ and $\{|f_i\rangle\}_{i=1}^{\infty} \subset H_b$ s.t*

$$|\psi\rangle = \sum_{i=1}^{\infty} \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle,$$

*where*

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$$l^1 = \{f : \mathbb{N} \to \mathbb{R} : \quad \sum_{n\in\mathbb{N}} |f(n)| < +\infty\}$$

Since the space of all real-valued integrable functions $L^1(X, \mathbb{R})$ are used in the theoretical discussion of problems in various field of science such as finance, engineering, physics, statistics, and other disciplines,we prefer to work more generally on $L^1(X, \mathbb{R})$ space. It is clear that for $\sigma$-finite measure space $\mathbb{N}$ equipped with the counting measure, $L^1$ and $l^1$ coincide.

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1. What is the Majorization?
2. Why $L^1$? (Quantum Interpretation)
3. Main Results
   - (Doubly)Stochastic Operators on $L^1(X, \mathbb{R})$
   - Majorization on $L^1(X, \mathbb{R}^n)$

**Definition 5** (*Stochastic Operator* or *Markov operator*)

Let $(X, \mu)$ and $(Y, \nu)$ be measure spaces such that $\mu(X) = \nu(Y)$. A linear operator $S : L^1(Y, \mathbb{R}) \to L^1(X, \mathbb{R})$ is called a *stochastic operator* if

1. $S$ is positive (that is, $S$ takes positive elements to positive elements), and
2. $\int_X Sf d\mu = \int_Y f d\nu, \quad \forall f \in L^1(Y, \mathbb{R})$.

Moreover, if in addition to the two conditions above, $\mu(X) = \nu(Y) < \infty$ and $S1 = 1$, then S is called a *doubly stochastic operator*.

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# (Doubly) Stochastic Integral Operator

**Definition 6 ((Doubly) Stochastic Kernel)**

A *stochastic kernel* is a measurable function $S : X \times Y \to [0,\infty)$ such that $\int_X S(x,y)d\mu(x) = 1$ for all $y \in Y$. A *doubly stochastic kernel* is a stochastic kernel with the additional property that $\int_Y S(x,y)d\nu(y) = 1$ for all $x \in X$.

**Definition 7 ((Doubly) Stochastic Integral Operator)**

An integral operator $M$ from $L^1(Y,\mathbb{R})$ to $L^1(X,\mathbb{R})$ given by $Mg = \int_Y S(x,y)g(y)d\nu(y)$ is said to be a *stochastic integral operator* (resp. *doubly stochastic integral operator*) if $S(x,y)$ is stochastic kernel (resp. doubly stochastic kernel).

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An integral operator $M$ from $L^1(Y, \mathbb{R})$ to $L^1(X, \mathbb{R})$ given by $Mg = \int_Y S(x, y) g(y) d\nu(y)$ is said to be a *stochastic integral operator* (resp. *doubly stochastic integral operator*) if $S(x, y)$ is stochastic kernel (resp. doubly stochastic kernel).

All (doubly) stochastic integral operators are (doubly) stochastic operators.

# Approximation by (Doubly) Stochastic Integral Operators

**Theorem 8**

*Let $(X, \mu)$ and $(Y, \nu)$ be $\sigma$-finite measure spaces. Let $S : L^1(Y) \to L^1(X)$ be a stochastic operator and let $V$ be a finite dimensional subspace of $L^1(Y)$. Then there exists a sequence of stochastic integral operators from $L^1(Y)$ to $L^1(X)$ which converge to $S$ on $V$.*

We also have an doubly stochastic version of this theorem:

**Theorem 9**

*Let $(X, \mu)$ and $(Y, \nu)$ be finite measure spaces. A doubly stochastic operator $D : L^1(Y, \mathbb{R}) \to L^1(X, \mathbb{R})$ on a finite dimensional subspace $V$ of $L^1(Y, \mathbb{R})$ can be approximated by doubly stochastic integral operators.*

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# What is $L^1(X,\mathbb{R}^n)$?

Given a measure space $(X,\mu)$, let $L^1(X,\mu,\mathbb{R}^n)$, or simply $L^1(X,\mathbb{R}^n)$, denote the set of all measurable functions $f$ from $(X,\mu)$ to $\mathbb{R}^n$ that satisfy $\int_X |f| d\mu < \infty$, where $|f|(x) = \sum_{k=1}^n |f_k(x)|$.

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# Matrix Majorization and Multivariate Majorization on $L^1(\cdot,\mathbb{R}^n)$

**Definition 10 (Matrix Majorization)**

If $f=(f_1,f_2,\dots,f_n)\in L^1(X,\mathbb{R}^n)$, $g=(g_1,g_2,\dots,g_n)\in L^1(Y,\mathbb{R}^n)$. Then we say that $f$ is matrix majorized by $g$, denoted $f\prec_M g$, if there exists a stochastic operator $S$ such that $f=S(g)$; i.e., $f_k=Sg_k$ for all $k=1,\dots,n$.

**Definition 11 (Multivariate Majorization)**

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# Matrix Majorization in Finite Dimensional Space

**Definition 12 (1999, Dahl-[3])**

Let $R \in M_{m\times n}(\mathbb{R})$ and $T \in M_{p\times n}(\mathbb{R})$ be two matrices. We say $R$ is *majorized* by $T$, denoted $R \prec T$ if there exists a column stochastic matrix $S \in M_{m\times p}(\mathbb{R})$ such that $R = ST$.

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# Our Terminology (Matrix Majorization)

Let $X = \{1, 2, ..., m\}$ and $\mu$ be counting measure. We can represent each function $f = (f_1, f_2, ..., f_n) \in L^1(X, \mathbb{R}^n)$ as an $m$ by $n$ non-negative matrix $M^f$ whose $k$th row is $f(k)$.

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## Theorem 13 (Theorem 3.3 in [3])

*Let $X=\{1,2,...,m\}$, $Y=\{1,2,...,p\}$, $f\in L^1(X,\mathbb{R}^n)$ and $g\in L^1(Y,\mathbb{R}^n)$. Then $f$ is matrix majorized by $g$ if and only if $\sum_{k=1}^m \phi(f(k)) \le \sum_{k=1}^p \phi(g(k))$ for all sublinear functional $\phi:\mathbb{R}^n\to\mathbb{R}$.*

This suggests the following one-side extension.

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*Let $(X,\mu)$ and $(Y,\nu)$ be two $\sigma$-finite measure spaces, $f\in L^1(X,\mathbb{R}^n)$ and $g\in L^1(Y,\mathbb{R}^n)$. If $f$ is matrix majorized by $g$, then for all sublinear functionals $\phi:\mathbb{R}^n\to\mathbb{R}$,*

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# Relations Between Matrix Majorization and Multivariate Majorization

In the setting of $\mathbb{R}^n$:

**Theorem 15**

*Let $(X, \mathcal{A}_1, \mu)$ and $(Y, \mathcal{A}_2, \nu)$ be finite measure spaces, $f \in L^1(X, \mathbb{R}^n)$, $g \in L^1(Y, \mathbb{R}^n)$, $h \in L^1(X, (0, \infty))$, and $k \in L^1(Y, (0, \infty))$. The following are equivalent:*

1. *$(f_1, f_2, \ldots, f_n, h)$ is matrix majorized by $(g_1, g_2, \ldots, g_n, k)$;*
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In the setting of $\mathbb{R}$:

**Theorem 16**

*Let $(X,\mu)$ and $(Y,\nu)$ be finite measure spaces, $f \in L^1(X,\mathbb{R})$, $g \in L^1(Y,\mathbb{R})$, $h \in L^1(X,(0,\infty))$, $k \in L^1(Y,(0,\infty))$. TFAE:*

1. *There exists a stochastic operator $S : L^1(Y,\mathbb{R},\nu) \to L^1(X,,\mathbb{R},\mu)$ such that $Sg = f$ and $Sk = h$.*
2. *For all real valued convex functions on $\mathbb{R}$,*
$$\int_X \phi\left(\frac{f}{h}\right) hd\mu \leq \int_Y \phi\left(\frac{g}{k}\right) kd\nu \text{ and } \int_X hd\mu = \int_Y kd\nu.$$
3. *There exists a doubly stochastic $D : L^1(Y,,\mathbb{R},\beta) \to L^1(X,\mathbb{R},\alpha)$ such that $D\left(\frac{g}{k}\right) = \frac{f}{h}$, where the measures $\alpha$ and $\beta$ are defined in earlier Theorem.*

In the setting of $\mathbb{R}$:

### Theorem 16

*Let $(X, \mu)$ and $(Y, \nu)$ be finite measure spaces, $f \in L^1(X, \mathbb{R})$, $g \in L^1(Y, \mathbb{R})$, $h \in L^1(X, (0, \infty))$, $k \in L^1(Y, (0, \infty))$. TFAE:*

1. *There exists a stochastic operator $S : L^1(Y, \mathbb{R}, \nu) \to L^1(X, , \mathbb{R}, \mu)$ such that $Sg = f$ and $Sk = h$.*
2. *For all real valued convex functions on $\mathbb{R}$,*
$$\int_X \phi\left(\frac{f}{h}\right) hd\mu \leq \int_Y \phi\left(\frac{g}{k}\right) kd\nu \text{ and } \int_X hd\mu = \int_Y kd\nu.$$
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***Thank You***
***For Your Attention***