Rigidity of Roe algebras

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joint work with Ján Špakula and Jiawen Zhang

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Summer School in Operator Algebras

Roe algebras

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- Let $\mathcal{H}_0$ be a Hilbert space. A bounded linear operator $T \in \mathfrak{B}(\ell^2(X; \mathcal{H}_0))$ can be written in the matrix form $\{T_{x,y}\}_{x,y \in X}$ where $T_{x,y} \in \mathfrak{B}(\mathcal{H}_0)$ defined by $T_{x,y}\xi = (T(\delta_y \otimes \xi))(x)$.

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Definition

- The **uniform Roe algebra** $C_u^*(X)$ is the norm closure of all finite propagation operators in $\mathfrak{B}(\ell^2(X; \mathbb{C}))$; E.g. $C_u^*(\Gamma) \cong \ell^\infty(\Gamma) \rtimes_r \Gamma$.

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- The **Roe algebra** $C^*(X)$ is the norm closure of all finite propagation and locally compact operators in $\mathfrak{B}(\ell^2(X;\ell^2(\mathbb{N})))$. E.g. $C^*(\Gamma)\cong\ell^\infty(\Gamma,\mathfrak{K}(\ell^2(\mathbb{N})))\rtimes_r\Gamma$.

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Proposition

If $X \sim_c Y$, then

1. $C^*(X) \cong C^*(Y)$;
2. $C^*_u(X) \otimes \mathfrak{K}(\ell^2(\mathbb{N})) \cong C^*_u(Y) \otimes \mathfrak{K}(\ell^2(\mathbb{N}))$ *(i.e., $C^*_u(X)$ and $C^*_u(Y)$ are **stably isomorphic**).*

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- The rigidity problems concern the opposite direction, *i.e.*, to what extent can Roe algebras determine the coarse geometry of the underlying spaces. More precisely, we ask the following:

Question

Let X, Y be metric spaces with bounded geometry.

(a) $C_u^*(X) \otimes \mathfrak{K}(\ell^2(\mathbb{N})) \cong C_u^*(Y) \otimes \mathfrak{K}(\ell^2(\mathbb{N})) \Rightarrow X \sim_c Y$?

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(b) $C^*(X) \cong C^*(Y) \Rightarrow X \sim_c Y$?

- (a)' is a weak form of rigidity.

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  - A subspace Y in a metric space (X,d) is **sparse** if $Y = \bigsqcup_n Y_n$ where each Y_n is finite and $d(Y_n, Y_m) \to \infty$ as $n+m \to \infty$ and $n \neq m$.

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Corollary

(a), (a)' and (b) hold if X or Y satisfies the coarse Baum-Connes conjecture ***with coefficients****.*

Analytic description

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$$P = \bigoplus_n P_n,$$

where each $P_n = (\cdot, \xi_n)\xi_n$ is a rank-one projection in $\mathfrak{B}(\ell^2(X_n; \mathcal{H}_0))$.

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- P is a ghost iff $\{(X_n, d_n, m_n)\}_n$ is **ghostly** (i.e. $\lim_n \sup_{x \in X_n} m_n(x) = 0$).

Theorem (L-Špakula-Zhang, 2020)

*Let $P \in \mathfrak{B}(\ell^2(X;\ell^2(\mathbb{N})))$ be a block-rank-one projection and m_n the associated measure on X_n. Then $P \in C^*(X)$ iff $\{(X_n, d_n, m_n)\}_n$ is a sequence of **measured asymptotic expanders**.*

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- When $c_\alpha \equiv c > 0$, we call it **Measured expanders**.
- When $c_\alpha \equiv c > 0$ and m_n = counting measure on finite graphs V_n, we recover **Expanders** for finite graphs $\{V_n\}_n$: $\exists\ c > 0\ \forall\ n$ and $\forall\ A \subset X_n$ with $0 < |A| \leq \frac{1}{2}|V_n|$, then $|\partial A| > c|A|$.

Outline of the proof

- Measured asymptotic expanders can be "nicely" approximated by measured expander graphs (V_n, E_n, m_n) with **bounded measure ratios** (i.e. If $u \sim_{E_n} v$ in V_n, then $s \cdot m_n(v) \leq m_n(u) \leq \frac{m_n(v)}{s}$ for some $0 < s < 1$).

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- The associated Laplacian operator $\Delta_n \in C^*(X)$ to (V_n, E_n, ν_n) has spectral gap at 0 in the spectrum. So $Q_n = \chi_{\{0\}}(\Delta_n) \in C^*(X)$ and $Q_n \to P$ up to a compact perturbation. Hence, $P \in C^*(X)$.

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Corollary

If either X or Y coarsely embeds into L^p-space for $p \in [1, \infty)$, then the rigidity (i.e., (a), (a)' and (b)) holds.

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Corollary (L-Špakula-Zhang, 2020)

There exist metric spaces that do ***not*** *coarsely embed into any L^p-space for $1 \leq p < \infty$, but the rigidity still holds.*

Thank you for your attention!