A non-commutative optimal coupling problem

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An "optimization" proof of the spectral theorem

Lemma

- If $A \in M_n(\mathbb{C})$, then $e^A = \sum_{k=0}^{\infty} (1/k!) A^k$ converges absolutely.
- We have $(e^A)^* = e^{A^*}$.
- If A and B commute, then $e^{A+B} = e^A e^B$.

Corollary

If $A \in M_n(\mathbb{C})_{sa}$, then $t \mapsto e^{itA}$ is a one-parameter group of unitaries, and it depends smoothly on t.

Observation

If D is a diagonal matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and A commutes with D, then $\lambda_i a_{i,j} = a_{i,j} \lambda_j$, hence A is diagonal. In particular, diagonal matrices are a maximal abelian subalgebra of $M_n(\mathbb{C})$.

An "optimization" proof of the spectral theorem

Theorem

If $X \in M_n(\mathbb{C})_{\mathrm{sa}}$, then there is a unitary U such that UXU^* is diagonal.

Proof.

Let Y be a diagonal matrix with real and distinct eigenvalues. Because the unitary group is compact, there exists a unitary matrix U that maximizes $\operatorname{tr}(UXU^*Y)$. If A is self-adjoint, then $\operatorname{tr}(e^{itA}UXU^*e^{-itA}Y)$ has a maximum at $t = 0$, and hence

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \operatorname{tr}(e^{itA}UXU^*e^{-itA}Y) \\ &= \operatorname{tr}(iAUXU^*Y) - \operatorname{tr}(UXU^*iAY) \\ &= \operatorname{tr}(Ai[UXU^*, Y]). \end{aligned}$$

Taking $A = i[UXU^*, Y]$, we obtain $[UXU^*, Y] = 0$. Since Y is diagonal with distinct eigenvalues, this implies that UXU^* is diagonal. □

Optimal couplings in matrix algebras

Let us denote by $L^2(M_n(\mathbb{C}), \mathrm{tr})$ the space of matrices with the inner product $\langle X, Y \rangle = \mathrm{tr}(X^* Y)$, where tr is the normalized trace (as in the GNS construction). If X and Y are self-adjoint and U is unitary, then

$$\begin{aligned} &\|UXU^* - Y\|_{L^2(M_n(\mathbb{C}),\mathrm{tr})}^2 \\ &\quad = \|U^*XU\|_{L^2(M_n(\mathbb{C}),\mathrm{tr})}^2 - 2\langle UXU^*, Y\rangle_{L^2(M_n(\mathbb{C}),\mathrm{tr})} + \|Y\|_{L^2(M_n(\mathbb{C}))}^2 \\ &\quad = \|X\|_{L^2(M_n(\mathbb{C}),\mathrm{tr})}^2 - 2\langle UXU^*, Y\rangle_{L^2(M_n(\mathbb{C}),\mathrm{tr})} + \|Y\|_{L^2(M_n(\mathbb{C}))}^2. \end{aligned}$$

Hence, minimizing $\|UXU^* - Y\|_{L^2(M_n(\mathbb{C}),\mathrm{tr})}$ over U is equivalent to maximizing $\langle UXU^*, Y\rangle_{L^2(M_n(\mathbb{C}),\mathrm{tr})}$.

The preceding argument shows that if U is a maximizer, then UXU^* and Y commute. In particular, if Y has distinct eigenvalues, then $UXU^* = f(Y)$ for some $f : \mathrm{Spec}(Y) \to \mathrm{Spec}(X)$. Assuming WLOG that Y is diagonal and considering $\Sigma UXU^* \Sigma^*$ for a permtutation matrix Σ, one can check that f must be increasing.

Optimal coupling for matrix tuples

Definition

For $X = (X_1, \dots, X_m) \in M_n(\mathbb{C})_{\text{sa}}^m$ and a unitary matrix U, let $UXU^* = (UX_1U^*, \dots, UX_mU^*)$. For $X, Y \in M_n(\mathbb{C})_{\text{sa}}^m$, let

$$C_n(X, Y) = \sup_{\text{unitaries } U} \langle UXU^*, Y \rangle_{L^2(M_n(\mathbb{C}), \text{tr})_{\text{sa}}^m}$$

over unitary matrices U.

We can argue that for a maximizer, $\sum_{j=1}^m [UX_jU^*, Y_j] = 0$. However, it is much less obvious how to describe the optimizers.

Optimal coupling for matrix tuples

Consider the inclusion $\iota : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) = M_{nk}(\mathbb{C})$. In the $m = 1$ case, the concrete description of the optimizers implies that

$$C_{nk}(\iota(X), \iota(Y)) = C_n(X, Y).$$

However, for $m > 1$, we will show that it is possible that

$$C_{nk}(\iota(X), \iota(Y)) > C_n(X, Y)$$

Although the optimal coupling problem for matrix tuples looks innocent, quantum information theory and the Connes embedding problem are lurking in the background!

Let's first discuss a more general framework for optimal couplings.

Optimal couplings in tracial von Neumann algebras

Recall that a tracial von Neumann algebra is a pair (A, τ), where A is von Neumann algebra and $\tau : A \to \mathbb{C}$ is a faithful normal tracial state.

Definition (Biane-Voiculescu 2001)

Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras, $X \in A_{\text{sa}}^m$ and $Y \in B_{\text{sa}}^m$ such that X and Y generate A and B respectively. We define

$$d_W(X, Y) = \inf \|\iota_1(X) - \iota_2(Y)\|_{L^2(M,\tau_M)_{\text{sa}}^m}$$
$$C(X, Y) = \sup \langle \iota_1(X), \iota_2(Y) \rangle_{L^2(M,\tau_M)_{\text{sa}}^m},$$

where the infimum and supremum are taken over tracial von Neumann algebras (M, τ_M) and embeddings (unital trace-preserving $*$-homomorphisms) $\iota_1 : (A, \tau_A) \to (M, \tau_M)$ and $\iota_2 : (B, \tau_B) \to (M, \tau_M)$.

d_W is called the *non-commutative Wasserstein distance*. The data $(M, \tau_M, \iota_1, \iota_2)$ describe a *coupling* of X and Y and the coupling is *optimal* if it achieves the infimum above.

Optimal couplings in tracial von Neumann algebras

Remark

If A is finite-dimensional and M is a factor, then any two embeddings of (A, τ_A) into (M, τ_M) are unitarily conjugate. In particular, if $A, B \subseteq M_n(\mathbb{C})$ and $M = M_n(\mathbb{C})$, then $\iota_j(Z) = U_j Z U_j^*$ for some unitary U_j, so we are in a similar situation to the previous discussion.

Remark

Similar to before, we have
$d_W(X, Y) = \|X\|^2_{L^2(A,\tau_A)^m_{\text{sa}}} - 2C(X, Y) + \|Y\|^2_{L^2(B,\tau_B)^m_{\text{sa}}}$.

Remark

The Wasserstein distance was introduced by Biane and Voiculescu as a free probabilistic analog of the Wasserstein distance for classical probability measures. The bulk of the GJNS paper investigates the optimal coupling problem from the viewpoint of classical optimal transport theory.

Optimal couplings and the Connes embedding problem

This talk will focus on the connection between the free probabilistic optimal coupling problem and quantum information theory.

Main proposition

The Connes embedding problem has a positive answer if and only if the following holds: For all $n, m \in \mathbb{N}$, for all $X, Y \in M_n(\mathbb{C})_{\mathrm{sa}}^m$, we have

$$\limsup_{k \to \infty} C_{nk}(\iota_k(X), \iota_k(Y)) = C(X, Y),$$

where ι_k is the canonical inclusion $M_n(\mathbb{C}) \to M_{nk}(\mathbb{C})$.

Thanks to the work of Ji-Natarajan-Vidick-Yuen-Wright, tentatively both of these conditions fail. The implication for free probability theory is that the failure of Connes embedding is an obstacle for convergence of the classical Wasserstein distance of random matrix models to the non-commutative Wasserstein distance.

Factorizable maps

To prove the proposition, it will be useful to reframe $C(X, Y)$ in a way that does not explicitly refer to (M, τ_M). Recall that if $\iota : (A, \tau_A) \to (M, \tau_M)$ is an embedding of tracial von Neumann algebras, then there is a unique corresponding trace-preserving conditional expectation $E : M \to A$. In fact, E is the adjoint of ι when considered as maps on the L^2 spaces.

Given $\iota_1 : (A, \tau_A) \to (M, \tau_M)$ and $\iota_2 : (B, \tau_B) \to (M, \tau_M)$, we can write

$$\langle \iota_1(X), \iota_2(Y) \rangle_{L^2(M,\tau_M)^m_{\text{sa}}} = \langle \iota_2^* \iota_1(X), Y \rangle_{L^2(B,\tau_B)^m_{\text{sa}}}.$$

Factorizable maps

Definition (Anantharaman-Delaroche)

Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras. A unital completely positive trace-preserving map $\Phi : (A, \tau_A) \to (B, \tau_B)$ is said to be *factorizable* if $\Phi = \iota_2^* \iota_1$ for some tracial von Neumann algebra embeddings $\iota_1 : (A, \tau_A) \to (M, \tau_M)$ and $\iota_2 : (B, \tau_B) \to (M, \tau_M)$, and in this case, we will say that Φ *factorizes through* (M, τ_M). We denote by $\mathrm{FM}(A, B)$ the space of factorizable maps (which does depend implicitly on the choice of traces).

Facts

$\mathrm{FM}(A, B)$ is a convex set that is compact in the pointwise weak-$*$ topology (B has a weak-$*$ topology since it is a von Neumann algebra). Also, factorizable maps between tracial von Neumann algebras are closed under composition.

Optimal couplings and factorizable maps

Observation

Let (A, τ_A) and (B, τ_B) be tracial von Neumann algebras, and let $X \in A_{sa}^m$ and $Y \in B_{sa}^m$. Then

$$C(X, Y) = \sup_{\Phi \in \mathrm{FM}(A,B)} \langle \Phi(X), Y \rangle_{L^2(B,\tau_B)_{sa}^m}.$$

Remark

Here we do not need to assume that X and Y generate A and B respectively. In the definition of $C(X, Y)$, we considered embeddings of $\mathrm{W}^*(X)$ and $\mathrm{W}^*(Y)$ into (M, τ_M). However, given a factorizable map $\mathrm{W}^*(X) \to \mathrm{W}^*(Y)$, one can compose it with the trace-preserving conditional expectation $A \to \mathrm{W}^*(X)$ and the embedding $\mathrm{W}^*(Y) \to B$ to obtain a factorizable map $A \to B$.

Key result of Haagerup and Musat

We will deduce our main proposition from the following result.

Theorem (Haagerup-Musat 2015)

A completely positive map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ factorizes through a Connes-embeddable tracial von Neumann algebra if and only if it is the limit of a sequence of maps $\Phi_j : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that factorize through $M_{nk_j}(\mathbb{C})$ for some j. Moreover, the Connes embedding problem has a positive answer if and only if, for every $n \in \mathbb{N}$, every factorizable map $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies these two equivalent conditions.

Based on this theorem, one direction of the main proposition is already clear. If the Connes embedding problem had a positive answer, then the von Neumann algebra (M, τ_M) associated to an optimal coupling would be Connes-embeddable, hence the corresponding factorizable map could be approximated by those that factorize through matrix algebras.

Proof of the main proposition

Conversely, suppose that Connes embedding problem has a negative answer. Let $\mathrm{FM}_{\mathrm{app}}(M_n(\mathbb{C}), M_n(\mathbb{C}))$ be the set of completely positive maps $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that factorize through a Connes-embeddable von Neumann algebra.

For $X, Y \in M_n(\mathbb{C})_{\mathrm{sa}}^m$, let

$$C_{\mathrm{app}}(X, Y) = \sup_{\Phi \in \mathrm{FM}_{\mathrm{app}}(M_n(\mathbb{C}), M_n(\mathbb{C}))} \langle \Phi(X), Y \rangle_{L^2(M_n(\mathbb{C}))_{\mathrm{sa}}^m}.$$

Based on Haagerup and Musat's theorem, we can see that

$$C_{\mathrm{app}}(X, Y) = \limsup_{k \to \infty} C_{nk}(\iota_k(X), \iota_k(Y)).$$

Our goal is to show that $C_{\mathrm{app}}(X, Y)$ is sometimes strictly less than $C(X, Y)$.

Proof of the main proposition

$\mathrm{FM}_{\mathrm{app}}(M_n(\mathbb{C}), M_n(\mathbb{C}))$ and $\mathrm{FM}(M_n(\mathbb{C}), M_n(\mathbb{C}))$ can be viewed as closed convex subsets of the space of real-linear maps $M_n(\mathbb{C})_{\mathrm{sa}} \to M_n(\mathbb{C})_{\mathrm{sa}}$, which is isomorphic to the space of real-linear maps $M_n(\mathbb{C})_{\mathrm{sa}} \otimes_{\mathbb{R}} M_n(\mathbb{C})_{\mathrm{sa}} \to \mathbb{R}$.

If $\Phi \in \mathrm{FM}(M_n(\mathbb{C}), M_n(\mathbb{C})) \setminus \mathrm{FM}_{\mathrm{app}}(M_n(\mathbb{C}), M_n(\mathbb{C}))$, then by the separating hyperplane theorem, there exists $v \in M_n(\mathbb{C})_{\mathrm{sa}} \otimes_{\mathbb{R}} M_n(\mathbb{C})_{\mathrm{sa}}$ such that

$$\Phi(v) > \sup_{\Psi \in \mathrm{FM}_{\mathrm{app}}(M_n(\mathbb{C}), M_n(\mathbb{C}))} \Psi(v).$$

Write v is a sum of simple tensors $v = \sum_{j=1}^m X_j \otimes Y_j$, and let $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_m)$. Then

$$\Phi(v) = \sum_{j=1}^m \langle \Phi(X_j), Y_j \rangle_{L^2(M_n(\mathbb{C}), \mathrm{tr})_{\mathrm{sa}}} = \langle \Phi(X), Y \rangle_{L^2(M_n(\mathbb{C}), \mathrm{tr})_{\mathrm{sa}}^m}.$$

Proof of the main proposition

Hence,

$$C(X, Y) \geq \langle \Phi(X), Y \rangle_{L^2(M_n(\mathbb{C}), \mathrm{tr})} > C_{\mathrm{app}}(X, Y).$$

This proves the proposition.

In other words, we have shown that given a negative answer to Connes embedding, there exist matrix tuples $X, Y \in M_n(\mathbb{C})_{\mathrm{sa}}^m$ for some n, m such that non-Connes-embeddable tracial von Neumann algebras are needed to witness the value of the Wasserstein distance. In particular, an optimal coupling can only occur in a non-Connes-embeddable von Neumann algebra.

Remark

Since the tensor rank of a vector in $V \otimes W$ is at most $\max(\dim V, \dim W)$, we see that in our argument we can take $m = n^2$.

The Connes-embeddable case

The Connes-embeddable version $C_{\mathrm{app}}(X, Y)$ is still somewhat badly behaved with respect to finite-dimensional approximation.

Proposition

For sufficiently large n, the following holds: For every $d \in \mathbb{N}$, there exist $X, Y \in M_n(\mathbb{C})_{sa}^{n^2}$ such that an optimal coupling requires a tracial von Neumann algebra with dimension at least d.

We rely on

Theorem (Musat-Rørdam 2020)

For large enough n, there exists a completely positive map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that factorizes through the hyperfinite II_1 factor but not through any finite-dimensional tracial von Neumann algebra.

We apply the hyperplane separation theorem to the closed convex set consisting of completely positive maps that factorize through algebras of dimension at most d.

Problems

Our paper does not give any explicit examples of optimals couplings for matrices.

Problem

Write a program to estimate $C_n(X, Y)$ for $X, Y \in M_n(\mathbb{C})_{sa}^m$ find approximate optimizers.

Problem

Give explicit examples of $X, Y \in M_n(\mathbb{C})_{sa}^m$ such that an optimal coupling requires an infinite-dimensional von Neumann algebra. (Similarly, for a coupling that is optimal among Connes-embeddable couplings.)

Problem

Use the optimal coupling problem as a strategy for finding more counterexamples to the Connes embedding problem.

Optimal coupling problem in terms of projections

To relate the optimal coupling problem more directly with tracial correlations between projections, consider $X, Y \in M_n(\mathbb{C})_{sa}^m$, and consider the spectral decompositions

$$X_j = \sum_{k=1}^n \lambda_{j,k} P_{j,k}, \qquad Y_j = \sum_{k=1}^n \mu_{j,k} Q_{j,k}.$$

Then $(P_{j,k})_{k=1}^m$ and $(Q_{j,k})_{k=1}^m$ are projection-valued measures. The optimal coupling problem asks us to maximize

$$\sum_{j=1}^m \sum_{k=1}^m \lambda_{j,k} \mu_{j,k} \tau_M(\iota_1(P_{j,k}) \iota_2(Q_{j,k}))$$

under the constraints that describe joint moments of the $P_{j,k}$'s and the joint moments of the $Q_{j,k}$'s.

Advertisement of further results

In the paper, we study an analog of the classical Monge-Kantorovich duality, using a new class of convex functions. As a consequence, we deduce the following result.

Theorem

If $(M, \tau_M, \iota_1, \iota_2)$ is an optimal coupling of X and Y and if $X' = \iota_1(X)$, $Y' = \iota_2(Y)$, then $\mathrm{W}^*((1-t)X' + tY') = \mathrm{W}^*(X', Y')$ for all $t \in (0,1)$.

Let $\Sigma_{m,R}$ be the space of equivalence classes of pairs (A, τ, X) where $X \in A_{\text{sa}}^m$, $A = \mathrm{W}^*(X)$, and $\|X\|_\infty \leq R$, where the equivalence relation is von Neumann algebra isomorphism respecting the generators and trace.

Proposition (Corollary of Gromov-Olshanskii-Ozawa theorem)

The space $\Sigma_{m,R}$ is not separable with respect to the Wasserstein distance d_W.

Advertisement of further results

The space $\Sigma_{m,R}$ has a natural weak-$*$ topology, the weakest topology such that the maps $(A, \tau, X) \mapsto \tau(p(X))$ are continuous for every non-commutative polynomial p. $\Sigma_{m,R}$ is compact and metrizable with respect to the weak-$*$ topology.

Proposition

The weak-$*$ topology and the Wasserstein topology agree at a point (A, τ, X) in $\Sigma_{m,R}$ if and only if every embedding of (A, τ) into a tracial ultraproduct lifts to a sequence of factorizable maps. In particular, if A is assumed to be Connes-embeddable, then the two topologies agree at (A, τ, X) if and only if A is amenable/AFD/semi-discrete/injective.

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