Von Neumann equivalence and group approximation properties

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Motivation

Questions (Boutonnet-Ioana-Peterson '18)

- *Is the class of properly proximal groups stable under measure equivalence?*
- *Is there a non-inner-amenable group which is not properly proximal?*

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$$\downarrow$$

A positive answer to the first question gives example of a group which is neither inner-amenable nor properly proximal!

Measure Equivalence

Definition (Gromov '93)

$\Gamma \overset{\mathrm{ME}}{\sim} \Lambda$, if there exists measurable, measure-preserving action $\Gamma \times \Lambda \curvearrowright (\Omega, m)$, and Borel subsets $Y, X \subset \Omega$ with $m(X), m(Y) < \infty$ so that

$$\Omega = \bigsqcup_{\gamma \in \Gamma} \gamma Y = \bigsqcup_{\lambda \in \Lambda} \lambda X.$$

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Example

Γ, Λ-lattices in a lcsc group G. Then $\Gamma \overset{\mathrm{ME}}{\sim} \Lambda$. $\Gamma \times \Lambda \curvearrowright G$:

$(\gamma, \lambda)g = \gamma g \lambda^{-1}$ preserves the Haar measure m_G.

ME, OE, and SOE

Theorem (Singer '55)

For free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ *and* $\Lambda \curvearrowright (Y, \nu)$, *the following are equivalent*

1. *There exists a* $*$*-isomorphism* $\Theta : L^\infty(X, \mu) \rtimes \Gamma \cong L^\infty(Y, \nu) \rtimes \Lambda$ *such that* $\Theta(L^\infty(X, \mu)) = L^\infty(Y, \nu)$.

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2. $\Gamma \curvearrowright (X, \mu) \overset{\mathrm{OE}}{\sim} \Lambda \curvearrowright (Y, \nu)$*, i.e., there exists a measure space isomorphism* $T : (X, \mu) \to (Y, \nu)$ *which takes* Γ*-orbits onto* Λ*-orbits.*

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Theorem (Ornstein-Weiss '80)

All countably infinite amenable discrete groups are measure equivalent.

ME invariants

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6. Vanishing of $H_b^2(\ell^2\Gamma)$ [Monod-Shalom '07]

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Theorem (Chifan-Ioana '11)

There exist two countable, discrete, ICC groups Γ and Λ which are orbit equivalent but not W^*-equivalent.

Von Neumann Equivalence

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Definition (Ishan-Peterson-Ruth '19)

A fundamental domain for $\Gamma \curvearrowright^{\sigma} \mathcal{M}$ is a projection $p \in \mathcal{M}$ such that $\{\sigma_\gamma(p)\}_{\gamma \in \Gamma}$ are pairwise orthogonal and $\sum_{\gamma \in \Gamma} \sigma_\gamma(p) = 1$.

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Definition (IPR '19)

$\Gamma \overset{\mathrm{vNE}}{\sim} \Lambda$ if there is a semifinite von Neumann algebra $(\mathcal{M}, \mathrm{Tr})$ with $\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \mathrm{Tr})$ such that each $\Gamma \curvearrowright \mathcal{M}$ and $\Lambda \curvearrowright \mathcal{M}$ has finite trace fundamental domains.

Examples

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$$\sigma_{(s,t)}(T) = \theta(\lambda_s)\rho_t T \rho_t^* \theta(\lambda_s^*),$$

where $\rho : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right regular representation. Rank one projection P_e onto the subspace $\mathbb{C}\delta_e$ is a common fundamental domain.

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where $\rho : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right regular representation. Rank one projection P_e onto the subspace $\mathbb{C}\delta_e$ is a common fundamental domain.
- If $\Gamma \curvearrowright (M_1, \tau_1)$ and $\Lambda \curvearrowright (M_2, \tau_2)$ and $\theta : M_1 \rtimes \Gamma \xrightarrow{\cong} M_2 \rtimes \Lambda$ with $\theta(M_1) = M_2$. Then $\Gamma \overset{\text{vNE}}{\sim} \Lambda$, and $\mathcal{M} = \langle M_1 \rtimes \Gamma, M_1 \rangle$ is a vN-coupling.

Proper Proximality

Definition (Boutonnet-Ioana-Peterson '18)

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- Convergence groups
- Non-amenable bi-exact groups
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- Lattices in non-compact semi-simple Lie groups of arbitrary rank

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Theorem (Boutonnet-Ioana-Peterson '18)

Properly proximal groups are not inner-amenable.

A non-inner-amenable, non-properly proximal group

Theorem (IPR '19)

Proper proximality is a vNE invariant. In particular, proper proximality is both an ME and W^*E invariant.

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Example (Duchesne, Tucker-Drob, Wesolek '18)

Class of inner-amenable groups is not closed under ME.

$$SL_3(\mathbb{F}_p[t^{-1}]) \ltimes \mathbb{F}_p[t, t^{-1}]^3 \quad \overset{\mathrm{ME}}{\sim} \quad SL_3(\mathbb{F}_p[t^{-1}] \ltimes \mathbb{F}_p[t^{-1}]^3) \times \mathbb{F}_p[t]^3$$

(not inner amenable) (inner amenable)

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Theorem (IPR '19)

Amenability, Haagerup property and Property (T) are vNE invariant.

Open Problems

- What other ME-invariants are vNE-invariants?
- Find examples of groups which are vNE but not ME.
- Develop the notion of vNE for locally compact groups.

Fin.