A Characterization of Frame-less Hilbert C^*-modules

Zahra Hasanpour-Yakhdani

University of Tehran

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Historical Background

Kasparov's stabilization theorem which plays an important role in Kasparov's KK-theory, asserts that every countably generated Hilbert C^*-module is a direct summand of $\oplus_{j\in\mathbb{N}} A_A$. Indeed, it shows that every countably generated Hilbert C^*-module admits a standard frame. It is a natural question to ask, whether this result can be generalized to an arbitrary Hilbert C^*-module? In other words, whether every Hilbert C^*-module admit a standard frame?

Frames in Hilbert Spaces

Frame in a Hilbert space H is defined as a family $\{(f_i)\}_{i\in I}$ of vectors of H with the property that there are constants $C, D > 0$ s.t.

$$C\|x\|^2 \leqslant \Sigma_{i\in I}|\langle x, f_i\rangle|^2 \leq D\|x\|^2.$$

In the case $C = D$, the frame is called tight frame. Also, if $C = D = 1$, then the frame is called normalized tight frame.

Frames in Hilbert C*-modules (M. Frank, D. R. Larson (2000))

Frames in Hilbert C^*-module is defines as a family $\{f_i\}_{i\in I}$ of vectors of Hilbert C*-module, such that $\Sigma_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ is convergent in unltra weak operator topology to some element in universal enveloping von-Neumann algebra of A. Also, there exist constants $0 < C \leq D < \infty$ such that for all $x \in X$,

$$C\langle x, x\rangle \leqslant \Sigma_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq D\langle x, x\rangle$$

In the case $C = D$, the frame is called tight frame. Also, if $C = D = 1$, then the frame is called normalized tight frame. In the case, the sum in the middle of the above inequality always converges in norm, then the frame $\{f_i\}$ is called a standard (normalized tight) frame.

Continuous Field of Hilbert Spaces

Continuous field of Hilbert spaces is defined as a pair $((H_t)_{t\in T}, \Gamma)$, where T is a locally compact Hausdorff space, $(H_t)_{t\in T}$ is a family of Hilbert spaces and Γ is a linear subspace of $\prod_{t\in T} H_t$ such that:

- For every $t \in T$, $\{x(t) | x \in \Gamma\} = H_t$;
- for every $x, x' \in \Gamma$ the map $t \mapsto \langle x(t), x'(t)\rangle_{H_t}$ is in $C_0(T)$;
- Γ is locally uniformly closed: if $x \in \prod_{t\in T} H_t$ and for each $\epsilon > 0$ and each $t \in T$, there is an $x' \in \Gamma$ such that $\|x(t') - x'(t')\| < \epsilon$ on a neighbourhood of t, then $x \in \Gamma$;

Structure of a Hilbert $C(T)$-module Admitting No Frames (H. Li(2010))

Let $((H_t)_{t\in T}, \Gamma)$ be a continuous field of Hilbert spaces over an infinite compact Hausdorff space T. There is a countable subset $W \subseteq T$ and a point $t_\infty \in \bar{W}/W$ that H_t is separable for every $t \in W$ and H_{t_∞} is non-separable. Moreover, Γ as a Hilbert $C(T)$-module has no frames.

Corollary (H. Li(2010))

Let A be a unital and commutative C*-algebra. Every Hilbert A-module admits a frame if and only if A is a finite dimensional C*-algebra.

Theorem (D. Bakić, B. Guljaš(2002))

Let A be a compact C*-algebra, i.e. admitting a non-degenerate representation into $K(H)$, for some Hilbert space H, then every Hilbert A-module X admits an orthonormal basis.

Conjecture (M. Amini, M. B. Asadi, G. A. Elliott, F. Khosravi(2017))

Every Hilbert C*-module over a C*-algebra A admits a frame if and only if A is a C^*-algebra of compact operators.

Holomorphic Hilbert Bundle of Dual Hopf Type (G. A. Elliott, K. Kawamura)

Let A be a C*-algebra. Denote by $P_0(A)$ the convex set of positive linear functionals on A of norm at most one (in other words, the set $P_0(A)$ of pure states of A together with the functional (0). Consider $P_0(A)$ with its natural uniform structure (determined by the seminorms arising from evaluation at the elements of A) and with its natural complex structure (determined by realizing each unitary equivalence class of pure states as the projective space based on the Hilbert space arising from one of them, of course the complex structure, is independent of the choice of representative pure state). The category of right Hilbert A-modules is equivalent to the category of uniform holomorphic Hilbert bundles over $p_0(A)$ of dual Hopf type.

Structure of a Hilbert A-module Admitting No Frame (M. B. Asadi, M. Frank, Z. Hassanpour-Yakhdani (2019))

Suppose that A is a C*-algebra, $f_0 \in P(A)$, $\pi_0 = [f_0]$, H_{π_0} is a separable Hilbert space and W is a countable subset of $P(A)$ such that $f_0 \in \overline{W} \setminus W$. If there exists a uniformly continuous holomorphic Hilbert bundle of dual Hopf type $\mathcal{H} = (B(H_\pi, K_\pi)e_\pi)_{(\pi,e_\pi)\in P_0(A)}$ such that for any $\pi \in [W]$, K_π is separable and K_{π_0} is nonseparable, then the Hilbert A-module $X(\mathcal{H})$ possess no frames.

A Hilbert A-module Admitting No Frames.(M. B. Asadi, M. Frank, Z. Hassanpour-Yakhdani (2019))

There exists a uniformly continuous holomorphic Hilbert bundle of dual Hopf type over $P(K(\ell^2) \dotplus \mathbb{C}I_{\ell^2})$ satisfying the conditions of last theorem. Hence, the C*-algebra $K(\ell^2) \dotplus \mathbb{C}I_{\ell^2}$ has a frame-less Hilbert module.

Theorem

Let A be a unital C*-algebra. Suppose that Hilbert A-module X admits a standard normalized tight frame, $\{f_i : i \in I\}$ for X.
Let

$$H_A = \{(a_i) : i \in I,\ \Sigma_i a_i a_i^* \text{ converges in } \|.\|_A\}.$$

Then the corresponding transform operator

$$\theta : X \to H_A, \qquad \theta(x) = \{(\langle x, f_i\rangle) : i \in I\}, \qquad x \in X$$

possesses an adjointable operator onto an orthogonal summand of H_A s.t. $\theta^*(e_i) = f_i$, for $e_i = (0_A, 0_A, ..., 1_{A,(i)}, 0_A, ...)$ and $i \in I$.
Moreover, the corresponding orthogonal projection $P : H_A \to \theta(X)$ fullfils $P(e_i) = \theta(f_i)$, for the standard orthogonal basis $\{e_i = (0_A, 0_A, ..., 1_{A,(i)}, 0_A, ...) : i \in I\}$ of H_A.
In particular, H_A is isomorphic to $X \bigoplus N$, where $N = \theta(X)^{\perp}$.

Theorem (Lj. Arambašić (2008))

For C*-algebra A, if there is a full Hilbert A-module X such that for every closed submodule M of X, $X = M \oplus N$ for some closed submodule N of X, then A is $*$-isomorphic to a C*-algebra of (not necessarily all) compact operators.

Conjecture

Let A be a unital C*-algebra that every Hilbert A-module X admits normalized tight frame. T.F.A.E. :

- For every closed submodule X of H_A, there is a closed submodule N of H_A , such that $M \oplus N$ is isomorphic to H_A.
- for every closed submodule X of H_A, there is a closed submodule N of H_A, such that $M \oplus N = H_A$.

If the conjecture holds then every Hilbert A-module admits a frame if and only if A is $*$-isomorphic to a finite dimensional C*-algebra.

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Thanks For Your Attention