Skew Products and Coactions for Topological Quivers

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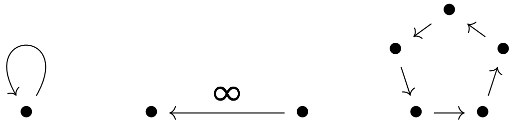
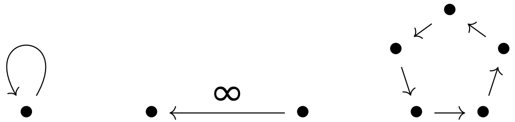


$$E = (E^0, E^1, r, s, \lambda)$$



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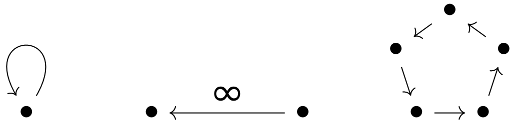
E-family $(p, s) : E \to B$ in C^*-algebra B

$$s_e{}^* s_e = p_{s(e)}, \quad s_e s_e{}^* \leq p_{r(e)}, \quad \sum_{r(e)=v} s_e s_e{}^* = p_v.$$



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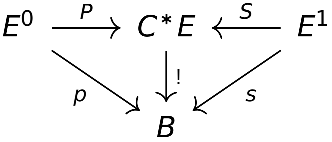

$E = (E^0, E^1, s, r, \lambda)$. $f \in c_0(E^0)$.

$$X = \bigoplus_0 \ell^2(s^{-1}(v), \lambda_v) \qquad xf(e) = x(e)f(s(e))$$

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$$\theta_{z,y}(x) = z\langle y, x\rangle; \quad \mathbb{K}(X) = \overline{\text{span}}\{\theta_{z,y}\}$$

Regular vertices :: $p_v = \sum \theta_{e,e}$

$$\begin{array}{ccc} c_0(\text{regular}) & \longrightarrow & \mathbb{K}(X) \\ \downarrow & & \downarrow \\ c_0(E^0) & \longrightarrow & \mathbb{L}(X) \end{array}$$

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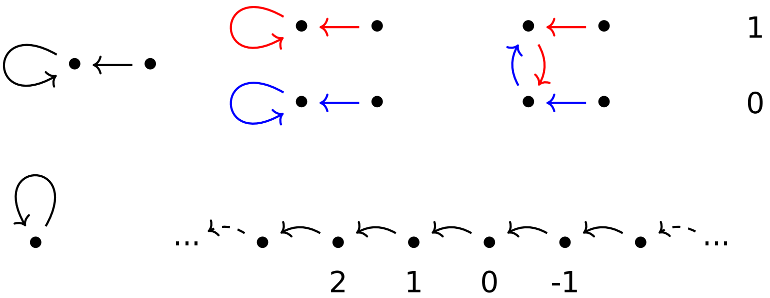
$$\begin{array}{ccccc} c_0(E^0) & \xrightarrow{k_A} & M(C^*E) & \xleftarrow{k_X} & X \\ & \searrow^{\pi} & \downarrow & \swarrow_{\psi} & \\ & & M(B) & & \end{array}$$

E graph, G discrete group. $c : E^1 \to G$ a *cocycle*.

$$E \times_c G = (E^0 \times G, E^1 \times G, \tilde{s}, \tilde{r}, \tilde{\lambda})$$
$$\tilde{s}(e,g) = (s(e), g), \quad \tilde{r}(e,g) = (r(e), c(e)g)$$



(A, G, α) an abelian C^*-dynamical system.
Covariant representation $(\pi, U) : (A, G, \alpha) \to B$

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^*$$

Crossed product $A \times_\alpha G$ universal for cov. reps.

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Takai-Takesaki Duality

$$A \times_\alpha G \times_\beta \Gamma \cong A \otimes \mathbb{K}(L^2(G))$$

$$(A \times_\alpha G, \Gamma, \beta) = (D, \hat{G}, \beta)$$

$$\tilde{\beta} : D \longrightarrow M(D \otimes C_0\hat{G})$$
$$d \longmapsto \beta.(d)$$

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Coaction (A, G, δ)

$$\delta : A \to M(A \otimes C^* G)$$

$$(\delta \otimes 1) \circ \delta = (1 \otimes \delta_G) \circ \delta$$

$$\delta(A) \cdot (1 \otimes C^* G) = A \otimes C^* G$$

$$\begin{array}{ccccc}
 & & A & \xrightarrow{\delta} & M(D \otimes C^* G) \\
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$A \times_\delta G$ coaction crossed product. Universal for covariant representations.

(A, G, α)	(A, G, δ)
$A \times_\alpha G$	$A \times_\delta G$
Dual Coaction	Dual Action
$\delta(U_g) = U_g \otimes g$	$\alpha(f) = f \circ \mathrm{rt}$
$\delta(a) = a \otimes 1$	$\alpha(a) = a \otimes 1$
Imai-Takai Duality	Katayama Duality
$A \times_\alpha G \times_\delta G \cong A \otimes \mathbb{K}$	$A \times_\delta G \times_\alpha G \cong A \otimes \mathbb{K}$

We've contrived a duality theory for nonabelian G.
What does it *look* like?
E graph, $c : E^1 \to G$ cocycle.
Induces a coaction δ on C^*E.

$$\begin{aligned}
\delta \circ k_A(f) &= k_A(f) \otimes 1 \\
\delta \circ k_X(fx) &= k_X(fx) \otimes c \\
&= (k_A(f) \otimes 1)(1 \otimes c)(k_X(x) \otimes 1) \\
&= (k_A(f) \otimes c)(k_X(x) \otimes 1)
\end{aligned}$$

$$C^*E \times_\delta G \cong C^*(E \times_c G)$$

Topological quiver $\mathcal{Q} = (E^0, E^1, s, r, \lambda)$.
$\lambda = \{\lambda_v\}_{v \in E^0}$, each full radon measure on $s^{-1}(v)$.
Continuous assembly: $f \in C_c(E^1)$, continuous assignment

$$v \mapsto \int f d\lambda_v$$

$$X = \Gamma_0(E^0; L^2(E^1, \lambda_v)) \quad \begin{aligned} fx(e) &= f(r(e))x(e) \\ xf(e) &= x(e)f(s(e)) \end{aligned}$$

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(X, A) is a C^*-correspondence. $C^*\mathcal{Q}$ universal C^*-algebra for covariant representations.

Skew Products for Quivers $c : E^1 \to G$ a *continuous* cocycle. $\mathcal{Q} \times_c G = (E^0 \times G, E^1 \times G, r_c, s_c, \lambda_c)$

$$s_c(e,g) = (s(e), g) \quad r_c(e,g) = (r(e), c(e)g)$$

$(X, A) = (X(\mathcal{Q}), C_0(E^0))$.

$$m : C_0(G) \to \mathbb{L}(X)$$
$$m(\gamma)(x)(e,g) = \gamma(c(e)g)x(e,g)$$

$$\begin{array}{ccccc}
 & & C^*\mathcal{Q} & \xrightarrow{\;\delta\;} & M(C^*\mathcal{Q} \otimes C^*G) \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi\otimes\mathrm{id}} \\
C_0G & \xrightarrow[\mu]{} & M(C^*(\mathcal{Q} \times_c G)) & \xrightarrow[\delta^\mu]{} & M(B \otimes C^*G)
\end{array}$$

$$\mathcal{Q} \times_c G = (E^0 \times G, E^1 \times G, \tilde{s}, \tilde{r}, \tilde{\lambda})$$

$$\begin{array}{ccccc} C^*\mathcal{Q} & \xrightarrow{j_{\mathcal{Q}}} & M(C^*\mathcal{Q} \times_\delta G) & \xleftarrow{j_G} & C_0(G) \\ & \searrow^{\pi} & \downarrow & \swarrow_{\mu} & \\ & & M(C^*(\mathcal{Q} \times_c G)) & & \end{array}$$

$C^*(\mathcal{Q}) \times_\delta G$ carries a dual action.
$C^*(\mathcal{Q} \times_c G)$ carries an action by right translation.
The *Dual Invariant Uniqueness Theorem* implies faithful.

Thanks!