Quantum Graphs

(joint work with Michael Brannan and Samuel Harris)

Priyanga Ganesan

Texas A&M University

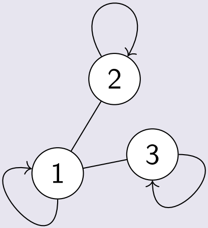
Summer School in Operator Algebras - 2021

Classical Graphs

G = (Vertex set, Edge set, Adjacency matrix)

Example of a reflexive graph

- $V = \{1, 2, 3\}$
- $E = \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$
- $A_G = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix}$



$$S_G := \left\{ \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix} \text{ where } * \in \mathbb{C} \right\} \subseteq M_3(\mathbb{C})$$

S_G is an operator system !

Non-commutative graph associated with a classical graph

Suppose G is a classical graph with vertex set $V = \{1, 2 \dots n\}$.

Graph Operator System

The non-commutative graph associated with the classical graph $G = (V, E)$ is the operator system S_G defined as

$$S_G = \text{ span}\{e_{ij} \ : \ (i,j) \in E \text{ or } i = j, \ \ \forall i, j \in V\} \subseteq \mathbb{M}_n \ ,$$

where e_{ij} are matrix units in $\mathbb{M}_n$.

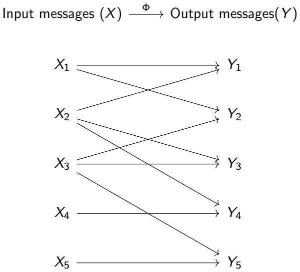
More generally,

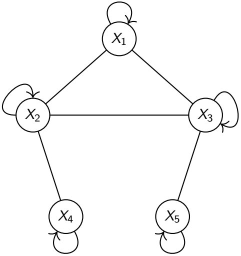
Matrix Quantum Graphs

An operator system in $\mathbb{M}_n$ is called a Matrix quantum graph.

Motivation from Information theory

- Used in zero-error communication.
- Generalize the confusability graph of classical channels.





A quantum channel is a completely-positive & trace-preserving linear map.

Non-commutative confusability graphs [DSW, 2010]

Given a quantum channel $\Phi : \mathbb{M}_m \to \mathbb{M}_n$ with $\Phi(x) = \sum_{i=1}^{r} K_i x K_i^*$, the confusability graph of Φ is the operator system:

$$S_\Phi = \text{ span}\{K_i^* K_j : 1 \leq i, j \leq r\} \subseteq \mathbb{M}_m.$$

Different approaches to quantum graphs

Classical graph $G = (V, E, A_G)$

- Quantize confusability graph of classical channels [DSW, 2010]
  - Matrix quantum graphs and Operator systems
  - Projection P_S onto the operator system S
- Quantize edge set $E \subseteq V \times V$ [Weaver, 2010, 2015]
  - Quantum relations
  - Projection P_E from χ_E
- Quantize adjacency matrix [MRV, 2018]
  - Categorical theory of quantum sets and quantum functions
  - Projection P_G using A_G

Unification

Under appropriate identifications, range of these projections is the same operator system!

Quantum graphs as quantum relations

Quantum set: von-Neumann algebra $\mathcal{M} \subseteq B(H)$.

Quantum relation [Weaver, 2010]

A quantum relation on $\mathcal{M}$ is a weak*-closed subspace $S \subseteq B(H)$ that is a bi-module over its commutant $\mathcal{M}'$.

Classical graph: $E \subseteq V \times V$ is a reflexive, symmetric relation on V.

Quantum Graph [Weaver, 2015]

A reflexive & symmetric quantum relation on $\mathcal{M}$ is a quantum graph.

$$ {}_{\mathcal{M}'}S_{\mathcal{M}'} \underset{\text{operator system}}{\subseteq} B(H) $$

The quantum adjacency matrix formalism of quantum graphs

Quantum set: finite dimensional C*-algebra $\mathcal{M}$ with a fixed tracial state.

Quantum graph with a quantum adjacency matrix [MRV, 2018]

A quantum graph is a pair $(\mathcal{M}, A_G)$ containing

- Quantum set $\mathcal{M}$
- Quantum adjacency matrix $A_G : \mathcal{M} \xrightarrow{\text{linear}} \mathcal{M}$ with
  - Idempotency: $m(A_G \otimes A_G)m^* = A_G$
  - Reflexivity: $m(A_G \otimes I)m^* = I$
  - Symmetry: $(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

where m is the multiplication map and σ is the swap map.

Comparing different notions of quantum graphs

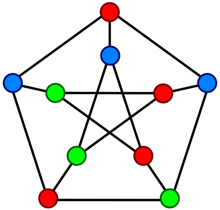
Quantum set M: finite dimensional C*-algebra with fixed tracial state ψ.

CLASSICAL GRAPH	MATRIX Q.GRAPH	AS QUANTUM RELATIONS	AS PROJECTIONS	WITH ADJACENCY MATRIX
$G = (V, E, A_G)$ $A_G \in \mathbb{M}_n\{0,1\}$	$S \subseteq \mathbb{M}_n$ is an operator system.	$(M, {}_{M'}S_{M'})$ weak*-closed operator sys in $B(H)$, bimodule over M'.	(M, p) $p \in M \otimes M^{op}$	(M, A_G) $A_G : M \to M$
Idempotency: $A_G \odot A_G = A_G$	$A_G \odot (\mathbb{M}_n) = S$	$M'SM' \subseteq S$	$p = p^* = p^2$	$m(A_G \otimes A_G)m^* = A_G$
Reflexivity: 1s on the diagonal	$1 \in S$	$M' \subseteq S$	$m(p) = 1_M$	$m(A_G \otimes I)m^* = I$
Undirected: $A_G = A_G^T$	$S = S^*$	$S = S^*$	$\sigma(p) = p$	$(\eta^* m \otimes I)(I \otimes A_G \otimes I)(I \otimes m^* \eta) = A_G$

Graph coloring

My research

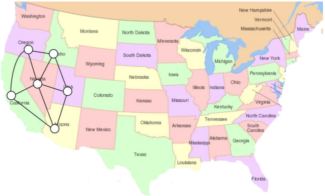
Assign colors to vertices of graph such that no adjacent vertices get same color.



Chromatic number

Least number of colors required to color that graph.

Quantum Coloring Problem

<table>
<tr><th></th><th>Classical
Graph</th><th>Quantum
Graph</th></tr>
<tr><td>Classical
Chromatic No.</td><td>
</td><td>Partition orthonormal basis into independent subsets</td></tr>
<tr><td>Quantum
Chromatic No.</td><td>

Non-Local Graph Coloring Game</td><td></td></tr>
</table>

The quantum-to-classical graph coloring game

1. Let $(S, \mathcal{M}, M_n)$ be a quantum graph.
2. $K_1 \oplus K_2 \oplus \ldots K_r = \mathbb{C}^n$ and $\mathcal{M}$ acts irreducibly on each K_j.
3. $\{v_1, v_2, \ldots v_n\} \underset{basis}{\subseteq} \mathbb{C}^n$ that can be partitioned into bases for $\{K_i\}_i^r$.

Definition (BGH, 2020)

The **quantum-to-classical graph coloring game** *for $(S, \mathcal{M}, M_n)$, with respect to the basis $\{v_1, ..., v_n\}$ and a quantum edge basis $\mathcal{F}$ for S is:*

- *Inputs:* $\sum_{p,q} y_{\alpha,pq} \;\; v_p \otimes v_q$, *where* $Y_\alpha := \sum_{p,q} y_{\alpha,pq} \;\; v_p v_q^* \in \mathcal{F}$.
- *Outputs:* *colors* $\{1, 2, \ldots k\}$.
- *Winning Criteria:*
  - *Adjacency rule: If $Y_\alpha \perp \mathcal{M}'$, then respond with different colors.*
  - *Same vertex rule: If $Y_\alpha \in \mathcal{M}'$, then respond with the same color.*

References

Runyao Duan, Simone Severini, and Andreas Winter.
Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number.
IEEE Trans. Inform. Theory, 59(2):1164–1174, 2013.

Benjamin Musto, David Reutter, and Dominic Verdon.
A compositional approach to quantum functions.
J. Math. Phys., 59(8):081706, 42, 2018.

Nik Weaver.
Quantum graphs as quantum relations, [arXiv:1506.03892].

Dan Stahlke.
Quantum zero-error source-channel coding and non-commutative graph theory.
IEEE Trans. Inform. Theory, 62(1):554–577, 2016.

Michael Brannan, Priyanga Ganesan, and Samuel J. Harris.
The quantum-to-classical graph homomorphism game, [arXiv:2009.07229].