# Lifts of completely positive equivariant maps

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based on joint work with Eusebio Gardella and Klaus Thomsen

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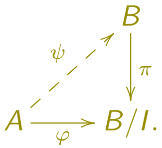

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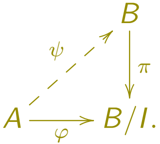


The importance of the existence of completely positive linear lifts can be traced back to Arveson's work on the extension theory.

Arveson considered the lifting problem for completely positive linear map $\varphi\colon C_c(X) \to \mathfrak{Q}(H)$, where $X$ is a compact metric space, $H$ is a separable Hilbert space, and $\mathfrak{Q}(H)$ is a Calkin algebra $(B(H)/K(H))$.

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Arveson used this result to give a simpler elegant proof of the Brown-Douglas-Fillmore theorem that $\mathrm{Ext}(C(X))$ is a group. In particular, Arveson showed that the availability of the lifting maps implies the existence of inverses in $\mathrm{Ext}(C(X))$.

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The Choi-Effros lifting theorem has numerous other applications in theory of $C^*$-algebras.

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Let's try to formulate an equivariant lifting problem:

1. $G$ is a locally compact, second countable group;
2. $(A, \alpha)$ and $(B, \beta)$ are $G$-algebras;
3. $I$ is a $G$-invariant ideal in $B$ with associated quotient map $\pi\colon B \to B/I$;
4. $\varphi\colon A \to B/I$ is equivariant completely positive contractive linear map.

$$\begin{array}{ccc} & & B \\ & \overset{\psi}{\nearrow} & \downarrow \pi \\ A & \xrightarrow[\varphi]{} & B/I. \end{array}$$

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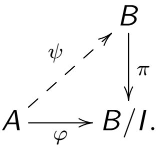


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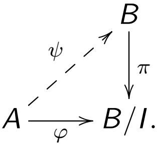


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Then $\psi$ is a $G$-equivariant completely positive linear lift for $\varphi$.

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Define $\psi\colon A \to B$ by $\psi(a) = \frac{1}{|F_G|} \Sigma_{g \in F_G} \beta_g(\rho(\alpha_{g^{-1}}(a)))$. Then for any $a \in F_A$ and $h \in K$, we get

$$\|\psi(\alpha_h(a)) - \beta_h(\psi(a))\| \leq \frac{2\|a\| |F_G \triangle k^{-1} F_G|}{|F_G|} \leq \varepsilon.$$

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2. For any $a \in F$,

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Instead of looking for an equivariant lift, we study the problem of finding "almost equivariant lift.

## Theorem 1 (F-Gardella-Thomsen)

Let $G$ be a second countable, locally compact group, let $(A, \alpha)$ and $(B, \beta)$ be $G$-algebras with $A$ separable.

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Given $\varepsilon > 0$, a finite subset $F_A \subseteq A$, a compact subset $K \subseteq G$ there exists a completely positive contractive linear map $\psi\colon A \to B$ with $\pi \circ \psi = \varphi$, satisfying the following conditions for all $g \in K$ and $a \in F_A$:

$$\|(\psi \circ \alpha_g)(a) - (\beta_g \circ \psi)(a)\| \leq \|(\varphi \circ \alpha_g)(a) - (\overline{\beta}_g \circ \varphi)(a)\| + \epsilon$$

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**Lemma 2**

Let $G$ be a locally compact, second countable group, let $(M, \gamma)$ be a $G$-algebra and let $I$ be a $\sigma$-unital $G$-invariant ideal in $M$. For every separable $C^*$-subalgebra $M_0 \subseteq M$, there exists a countable approximate unit $(x_n)_{n\in\mathbb{N}}$ in $I$ satisfying the following conditions

(a) $x_n \leq x_{n+1}$ and $x_{n+1}x_n = x_n$ for all $n \in \mathbb{N}$;

(b) $\lim_{n\to\infty} \|x_n b - bx_n\| = 0$ for all $b \in M_0$;

(c) $\lim_{n\to\infty} \max_{g\in K} \|\gamma_g(x_n) - x_n\| = 0$ for all compact subsets $K \subseteq G$.

## Notation

Let $(I, \gamma)$ be a $G$-algebra, and let $M(I)$ be the multiplier algebra of $I$.

For each $g \in G$, the automorphism $\gamma_g \in \mathrm{Aut}(I)$ extends to an automorphism $\widetilde{\gamma}_g$ of $M(I)$, and the resulting assignment $\widetilde{\gamma}\colon G \to \mathrm{Aut}(M(I))$ is a group homomorphism.

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Since $I$ is invariant by $\widetilde{\gamma}$, it follows that there is also a group homomorphism $\overline{\gamma} \colon G \to \mathrm{Aut}(Q(I))$ such that $\overline{\gamma}_g \circ q_I = q_I \circ \widetilde{\gamma}_g$ for all $g \in G$.

## Proposition 3

Let $G$ be a second countable, locally compact group, let $(A, \alpha)$ and $(I, \gamma)$ be $G$-algebras, and assume that $A$ is separable and $I$ is $\sigma$-unital. Let $\varphi\colon A \to Q(I)$ and $\theta\colon A \to M(I)_{\widetilde{\gamma}}$ be completely positive contractions such that $q_I \circ \theta = \varphi$.

Given a finite subset $F \subseteq A$, a compact subset $K_G \subseteq G$, and $\varepsilon > 0$, there exists a completely positive contraction $\theta'\colon A \to M(I)_{\widetilde{\gamma}}$ with $q_I \circ \theta' = \varphi$, satisfying the following conditions for all $a \in F$:

$$\max_{g \in K_G} \left\|\widetilde{\gamma}_g(\theta'(a)) - \theta'(\alpha_g(a))\right\| \leq \max_{g \in K_G} \left\|\overline{\gamma}_g(\varphi(a)) - \varphi(\alpha_g(a))\right\| + \epsilon. \tag{1}$$

## The idea of the proof of Proposition 2:

- Let $M_0 \subseteq M(I)_{\bar{\gamma}}$ be the separable $G$-algebra generated by $\theta(A)$. Let $(x_n)_{n\in\mathbb{N}}$ be an approximate identity for $I$ as in the conclusion of Lemma 1 for $M_0$.
- Set $\Delta_0 = \sqrt{x_0}$ and $\Delta_n = \sqrt{x_n - x_{n-1}}$ for all $n \geq 1$.

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- Each $\theta_m$ is a lift for $\varphi$. The choice of $(x_n)$ and the properties of $\Delta_n$ enable us to choose $m$ such that $\theta_m$ satisfies in (1) for the given $F_A \subseteq A$, $K \subseteq G$ and $\varepsilon > 0$

The we use the Busby invariant for equivariant extensions of $G$-algebras to translate between the statement of Theorem 1 and that of Proposition 2.

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The question is that how we can reformulate Theorem 1 without the assumption of the existence a completely positive linear lift. Indeed, we show that there is a continuous family $\Theta = (\Theta_t)_{t\in[1,\infty)} \colon A \to B$ of lifts of $\varphi$, which are asymptotically linear, asymptotically completely positive, and asymptotically equivariant.

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(a) *unital*, if $\Theta_t(1) = 1$ for all $t$;

(b) *self-adjoint*, if $\Theta_t$ is self-adjoint for all $t \in [1,\infty)$;

(c) *asymptotically linear*, if

$$\lim_{t\to\infty} \|\Theta_t(\lambda_1 a_1 + \lambda_2 a_2) - \lambda_1\Theta_t(a_1) - \lambda_2\Theta_t(a_2)\| = 0$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $a_1, a_2 \in A$;

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(f) *asymptotically completely positive*, if the continuous family

$$\Theta \otimes \mathrm{id}_{M_n} = (\Theta_t \otimes \mathrm{id}_{M_n})_{t\in[1,\infty)} : M_n(A) \to M_n(B)$$

is asymptotically linear and asymptotically positive for all $n$.

**Definition.** Let $(A, \alpha)$ and $(B, \beta)$ be are $G$-algebras and $I$ be a $G$-invariant ideal of $B$ with the quotient map $q\colon A \to B/I$. Let $\psi\colon A \to B/I$ be a linear completely positive contractive map, and let $\Theta = (\Theta_t)_{t\in[1,\infty)}\colon A \to B$ be a continuous path of map.

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**Theorem 4 (F-Gardella-Thomsen)**

Let $G$ be a second countable locally compact group, $(A, \alpha)$ and $(B, \beta)$ be are $G$-algebras and $I$ be a $G$-invariant ideal of $B$. Suppose that $A$ is separable and $\psi \colon A \to B/I$ is a linear completely positive contractive map. There is an asymptotically equivariant lift $\Theta = (\Theta_t)_{t\in[1,\infty)} \colon A \to B$ of $\psi$,

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(a) for $a, a' \in A$, we have $\lim_{t\to\infty} \|\Theta_t(a)\Theta_t(a)')\| = \|\psi(a)\psi(a')\|$.

(b) if $I$ is $\sigma$-unital, then $\lim_{t\to\infty} \Theta_t(a)x = 0$ for all $a \in A$ and $x \in I$.

**Theorem 5 (F-Gardella-Thomsen)**

Let $G$ be a second countable locally compact group, $(A, \alpha)$ and $(B, \beta)$ be are unital $G$-algebras and $I$ be a $G$-invariant $\sigma$-ideal of $B$ with the quotient map $q \colon A \to B/I$. Suppose that $A$ is separable with $G$-invariant state $\chi$ and $\psi \colon A \to B/I$ is a unital linear completely positive contractive map. There is a unital asymptotically equivariant lift $\Theta = (\Theta_t)_{t \in [1,\infty)} \colon A \to B$ of $\psi$.

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2. Every unital separable $G$-algebra $(B, \beta)$ has a $G$-invariant state.
3. $G$ is amenable.

*Thank you for listening!*