Free group factor problem and Popa's MV Property

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June 4, 2021

Summer School in Operator Algebras

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von Neumann Algebras

$B(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H} \mid$ linear, bounded$\}$

$$\|T\|_\infty = \sup_{\|\xi\| \leq 1} \|T(\xi)\| < \infty$$

Strong Operator Topology (SOT) $T_i \xrightarrow{SOT} T$ if and only if $\|T_i(\xi) - T(\xi)\| \to 0$.

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Theorem (Von Neumann Bicommutant Theorem '26)

Let $1 \in \mathcal{M} \subseteq B(\mathcal{H})$ be a $*$-subalgebra. Then $\overline{\mathcal{M}}^{SOT} = \mathcal{M}''$.
(Here, when $X \subseteq B(\mathcal{H})$ we denote by
$X' := X' \cap B(\mathcal{H}) = \{T \in B(\mathcal{H}) \mid xT = Tx \quad \forall x \in X\}$)

von Neumann algebras

Definition

A $*$-subalgebra $1 \in \mathcal{M} \subseteq B(\mathcal{H})$ is called a von Neumann algebra if $\mathcal{M} = \overline{\mathcal{M}}^{SOT} (= \mathcal{M}'')$.

Examples: $B(\mathcal{H})$; X' for every subset $X \subseteq B(\mathcal{H})$; $L^\infty([0,1]) \subset B(L^2[0,1])$

A von Neumann algebra $\mathcal{M}$ is called a factor if $\mathcal{Z}(\mathcal{M}) = \mathbb{C}$

Group von Neumann algebras

- (Murray-von Neumann '43)
- Γ - countable discrete group

⇝ $u : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ - left regular representation

$$u_\gamma(\xi)(\lambda) = \xi(\gamma^{-1}\lambda), \quad \forall \gamma, \lambda \in \Gamma, \xi \in \ell^2\Gamma$$

⇝ the von Neumann algebra associated with Γ is

$$\mathcal{L}(\Gamma) := \{u_\gamma \mid \gamma \in \Gamma\}'' = \overline{\mathbb{C}[\Gamma]}^{SOT} \subset \mathfrak{B}(\ell^2\Gamma)$$

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⇝ $\tau(x) = \langle x\delta_e, \delta_e \rangle$ normal, state

- (faithful) $\tau(x^*x) = 0 \Leftrightarrow x = 0$
- (tracial) $\tau(xy) = \tau(yx)$

⇝ $\mathcal{L}(\Gamma)$ is a finite von Neumann algebra $(v^*v = 1 \Rightarrow vv^* = 1)$

Group von Neumann Algebras

Definition

$\mathcal{M} \subseteq \mathbb{B}(\mathcal{H})$ is called a II_1 factor if $\mathcal{M}$ is an infinite dimensional factor, and admits a faithful, normal trace.

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Theorem (Murray-von Neumann '43)

$\mathcal{L}(\Gamma)$ is a II_1 factor ($\mathcal{Z}(\mathcal{L}(\Gamma)) = \mathbb{C}1$) $\Leftrightarrow \forall\ \gamma \neq e$ we have $|\gamma^\Gamma| = \infty$, i.e. Γ is icc.

Examples:

- $\mathcal{S}_\infty$; $\mathbb{Z} \wr \mathbb{Z}$; $\mathbb{Z}_2 \wr \mathbb{Z}$;
- $\mathbb{F}_n$, $n \geq 2$; $\Gamma_1 * \Gamma_2$, $|\Gamma_1| \geq 2, |\Gamma_2| \geq 3$;

Distinguishing von Neumann algebras

- (M-vN '43) $\mathcal{L}(\mathcal{S}_\infty) \cong \mathcal{L}(\mathcal{A}_\infty) \cong \mathcal{L}(H)$, for any locally finite i.c.c. group H.
- (Connes '76) $\mathcal{L}(\mathcal{S}_\infty) \cong \mathcal{L}(H) \cong \mathcal{R}$, for any amenable, i.c.c. group H.

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- (Connes '76) $\mathcal{L}(\mathcal{S}_\infty) \cong \mathcal{L}(H) \cong \mathcal{R}$, for any amenable, i.c.c. group H.
- (M-vN '43) $\mathcal{L}(\mathcal{S}_\infty) \ncong \mathcal{L}(\mathbb{F}_n)$, $n \geq 2$.

Free group factor problem

Question

Is $\mathcal{L}(\mathbb{F}_2) \cong \mathcal{L}(\mathbb{F}_\infty)$?

II_1 Factors

Theorem

Let $\mathcal{M}$ be a II_1 factor with trace τ. Then $\{\tau(p) : p \in \mathcal{P}(\mathcal{M})\} = [0,1]$.

Definition

Let $\mathcal{M}$ be a II_1 factor with trace τ.

- Let $n \in \mathbb{N}$. Then $\mathcal{M}^n := \mathcal{M}\bar{\otimes}\mathbb{M}_n(\mathbb{C}) = \mathbb{M}_n(\mathcal{M})$.
- Note that $\mathbb{M}_n(\mathcal{M})$ is a II_1 factor with $\tau_n([x_{i,j}]) = \frac{1}{n}\sum_{i=1}^n \tau(x_{i,i})$.

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- Note that $\mathbb{M}_n(\mathcal{M})$ is a II_1 factor with $\tau_n([x_{i,j}]) = \frac{1}{n}\sum_{i=1}^n \tau(x_{i,i})$.
- Let $1 < t < n$. Then $\mathcal{M}^t = p(\mathcal{M}\bar{\otimes}\mathbb{M}_n(\mathbb{C}))p$ where $p \in \mathcal{P}(\mathbb{M}_n(\mathcal{M}))$ with $\tau_n(p) = t/n$.
- Let $0 < t < 1$. Then $\mathcal{M}^t = p\mathcal{M}p$ where $p \in \mathcal{P}(\mathcal{M})$ with $\tau(p) = t$.
- Let $\mathcal{M}$ be a II_1 factor. Then $\mathcal{F}(\mathcal{M}) = \{t \in \mathbb{R}_+ : \mathcal{M}^t \cong \mathcal{M}\}$

Free group factor problem

Theorem (Voiculescu 1989, Radulescu 1991)

$\mathcal{F}(\mathcal{L}(\mathbb{F}_\infty)) = \mathbb{R}_+$

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Free group factor problem

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Theorem (Voiculescu 1989, Radulescu 1991, Dykema 1992)

$\mathcal{F}(\mathcal{L}(\mathbb{F}_2)) \in \{\mathbb{R}_+, \{1\}\}$.

Moreover, if $\mathcal{F}(\mathcal{L}(\mathbb{F}_2)) = \{1\}$, *then* $\mathcal{L}(\mathbb{F}_2) \ncong \mathcal{L}(\mathbb{F}_\infty) \ncong \mathcal{L}(\mathbb{F}_n)$, *for all* $n \geq 2$.

If $\mathcal{F}(\mathcal{L}(\mathbb{F}_2)) = \mathbb{R}_+$, *then* $\mathcal{L}(\mathbb{F}_2) \cong \mathcal{L}(\mathbb{F}_\infty) \cong \mathcal{L}(\mathbb{F}_n)$, *for all* $n \geq 2$.

Popa's Strategy for showing $\mathcal{L}(\mathbb{F}_2) \ncong \mathcal{L}(\mathbb{F}_\infty)$

- A II_1 factor $\mathcal{M}$ is called *stably singly generated*(SSG) if $\mathcal{M}^t$ is singly generated for all $t > 0$.

Popa's Strategy for showing $\mathcal{L}(\mathbb{F}_2) \not\cong \mathcal{L}(\mathbb{F}_\infty)$

- A II_1 factor $\mathcal{M}$ is called *stably singly generated*(SSG) if $\mathcal{M}^t$ is singly generated for all $t > 0$.
- If a II_1 factor $\mathcal{M}$ has nontrivial fundamental group, and is finitely generated, then $\mathcal{M}$ is stably singly generated.
- Thus, if $\mathcal{L}(\mathbb{F}_\infty) \cong \mathcal{L}(\mathbb{F}_2)$, then $\mathcal{L}(\mathbb{F}_\infty)$ is stably singly generated.

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- Thus, if $\mathcal{L}(\mathbb{F}_\infty) \cong \mathcal{L}(\mathbb{F}_2)$, then $\mathcal{L}(\mathbb{F}_\infty)$ is stably singly generated.
- Popa conjectures that any SSG factor $\mathcal{M}$ is *tight*, i.e. $\mathcal{M}$ has two hyperfinite subfactors $\mathcal{R}_0$ and $\mathcal{R}_1$ such that $\mathcal{R}_0 \vee \mathcal{R}_1^{op} = \mathcal{B}(L^2(\mathcal{M}))$.

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- Thus, if $\mathcal{L}(\mathbb{F}_\infty) \cong \mathcal{L}(\mathbb{F}_2)$, then $\mathcal{L}(\mathbb{F}_\infty)$ is stably singly generated.
- Popa conjectures that any SSG factor $\mathcal{M}$ is *tight*, i.e. $\mathcal{M}$ has two hyperfinite subfactors $\mathcal{R}_0$ and $\mathcal{R}_1$ such that $\mathcal{R}_0 \vee \mathcal{R}_1^{op} = \mathcal{B}(L^2(\mathcal{M}))$.
- However, $\mathcal{L}(\mathbb{F}_\infty)$ isn't tight by [Ge-Popa 1996].

Popa's MV-property

Popa observed that any tight factor satisfies the MV-property.

Definition (Popa 2019)

Let $\mathcal{M}$ be a II_1 factor. Then $\mathcal{M}$ has the Mean Value property (MV-property) if for all $T \in \mathcal{B}(L^2(\mathcal{M}))$ the weak closure of the convex hull of $uv^{op}Tv^{op*}u^*$ intersects $\mathbb{C}$, where u and v run over all unitaries in $\mathcal{M}$.

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Question (Popa 2019)

Does $\mathcal{L}(\mathbb{F}_2)$ satisfy the MV-property?

Free group factors have the MV-property

Theorem (D-Peterson 2019)

$\mathcal{L}(\mathbb{F}_n)$ have the MV-property for all $n \geq 2$.

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The proof uses noncommutative Poisson boundaries.

Noncommutative Poisson boundary of $\mathcal{L}(\mathbb{F}_2)$

Let μ be the probability measure on $\mathbb{F}_2 = \langle a, b \rangle$ defined by $\mu(a) = \mu(a^{-1}) = \mu(b) = \mu(b^{-1}) = \frac{1}{4}$. Consider the (Markov) operator $\mathcal{P}_\mu : \mathcal{B}(\ell^2(\mathbb{F}_2)) \to \mathcal{B}(\ell^2(\mathbb{F}_2))$ given by

$$\mathcal{P}_\mu(T) = \sum_{g \in \mathcal{S}} \mu(g) \rho_g T \rho_g^*,$$

where $\mathcal{S} = \{a, a^{-1}, b, b^{-1}\}$, and ρ denotes the right regular representation.

Noncommutative Poisson boundary of $\mathcal{L}(\mathbb{F}_2)$ continued

$\mathcal{P}_\mu(T) = \sum_{g \in \mathcal{S}} \mu(g) \rho_g T \rho_g^*$.

Let $\mathrm{Har}(\mathcal{P}_\mu) = \{\mathrm{T} \in \mathcal{B}(\ell^2(\mathbb{F}_2)) : \mathcal{P}_\mu(\mathrm{T}) = \mathrm{T}\}$.

Then $\mathrm{Har}(\mathcal{P}_\mu)$ is a weakly closed, injective operator system,

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Noncommutative Poisson boundaries for $\mathcal{L}(\Gamma)$, where Γ is an i.c.c. group, were studied by Izumi (2000s), Peterson-Creutz (2012).

Proof outline for $\mathcal{L}(\mathbb{F}_2)$

Let $\mathcal{P}_\mu(T) = \sum_{g\in\mathcal{S}} \mu(g)\rho_g T\rho_g^*$,

Let $\mathcal{P}_\mu^o(T) = \sum_{g\in\mathcal{S}} \mu(g)\lambda_g T\lambda_g^*$, where λ denotes the left regular representation

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Theorem (Double Ergodicity Theorem, D-Peterson 2019)

Let $T \in \mathcal{B}(\ell^2(\mathbb{F}_2))$ be such that $\mathcal{P}_\mu(T) = \mathcal{P}_\mu^o(T) = T$. Then $T \in \mathbb{C}$.

Proof outline for $\mathcal{L}(\mathbb{F}_2)$ continued

To show that $\mathcal{L}(\mathbb{F}_2)$ has the MV-property we proceed as follows:

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To show that $\mathcal{L}(\mathbb{F}_2)$ has the MV-property we proceed as follows:

Let $T \in \mathcal{B}(\ell^2(\mathbb{F}_2))$.

Let S be a weak operator topology limit point of $\{\frac{1}{N}\sum_{n=1}^{N} \mathcal{P}_{\mu}^{n}(T)\}_N$

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Then $\mathcal{P}_\mu(S) = S$.

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Then $\mathcal{P}_\mu(S) = S$.

Let R be a weak operator topology limit point of $\{\frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\mu^o)^n(S)\}_N$

Then $\mathcal{P}_\mu^o(R) = R$.

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Let R be a weak operator topology limit point of $\{\frac{1}{N}\sum_{n=1}^{N}(\mathcal{P}_\mu^o)^n(S)\}_N$

Then $\mathcal{P}_\mu^o(R) = R$.

Note that $\mathcal{P}_\mu \circ \mathcal{P}_\mu^o = \mathcal{P}_\mu^o \circ \mathcal{P}_\mu$. Thus, $\mathcal{P}_\mu^o(R) = R = \mathcal{P}_\mu(R)$, as $\mathcal{P}_\mu$ is normal.

Proof outline for $\mathcal{L}(\mathbb{F}_2)$ continued

By the Double Ergodicity Theorem, $\mathcal{P}^o_\mu(R) = R = \mathcal{P}_\mu(R)$, implies that $R \in \mathbb{C}$.

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By the Double Ergodicity Theorem, $\mathcal{P}_{\mu}^{o}(R) = R = \mathcal{P}_{\mu}(R)$, implies that $R \in \mathbb{C}$.

As $\mathcal{P}_{\mu}(T) = \sum_{g \in \mathcal{S}} \mu(g) \rho_g T \rho_g^*$, and $\mathcal{P}_{\mu}^{o}(T) = \sum_{g \in \mathcal{S}} \mu(g) \lambda_g T \lambda_g^*$, we get that $\mathcal{L}(\mathbb{F}_2)$ has the MV-property.

MV- property for any II_1 factor $\mathcal{M}$

Theorem (D-Peterson)

Let $\mathcal{M}$ be any II_1 factor. Then $\mathcal{M}$ satisfies the MV-property.

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Question (Popa 2019)

If $\mathcal{M}$ is a SSG factor, then does there exist a hyperfinite subfactor $\mathcal{R}$ of $\mathcal{M}$ such that $\mathcal{M} \subseteq \langle \mathcal{M}, e_{\mathcal{R}} \rangle$ is MV-ergodic?

Thank You

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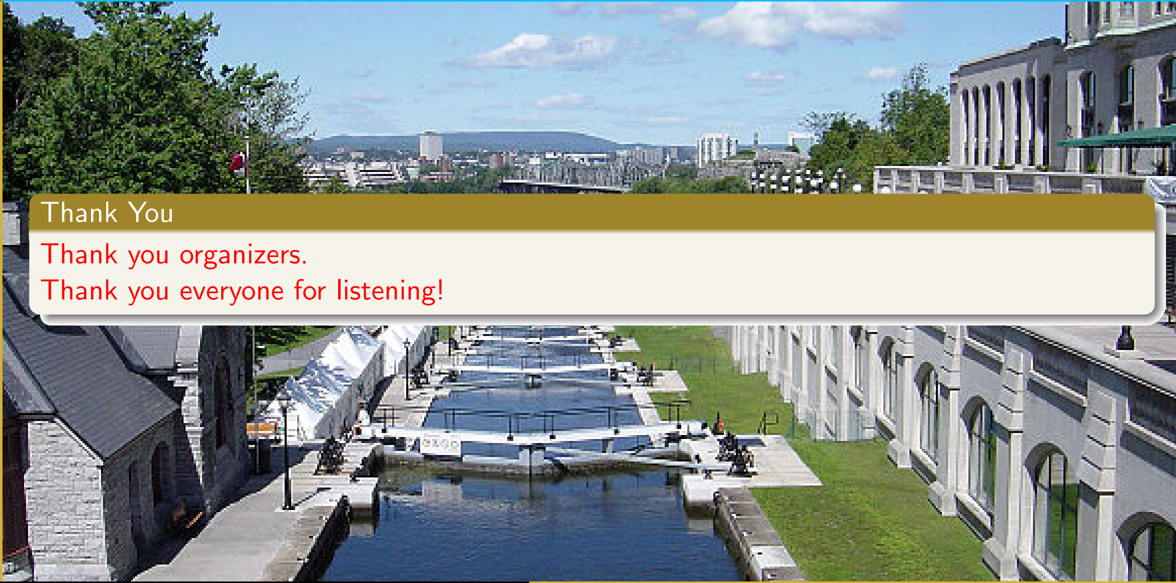

Thank You

Thank you organizers.

Thank you everyone for listening!