Conjugacy of local homeomorphisms

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Overview

Spoiler: main result

Theorem (Armstrong–B–Carlsen–Eilers)

Let (X, σ) and (Y, τ) be (partial) local homeomorphisms on locally compact Hausdorff spaces. They are conjugate if and only if there is a *-isomorphism $\varphi\colon C^*(X, \sigma) \to C^*(Y, \tau)$ satisfying

$$\varphi \circ \gamma_t^{X, g\circ h} = \gamma_t^{Y,g} \circ \varphi, \qquad g \in C(Y, \mathbb{R}), t \in \mathbb{R},$$

for some homeomorphism $h\colon X \to Y$.

Conjugacy means: there exists a homeomorphism $h\colon X \to Y$ such that $h \circ \sigma = \tau \circ h$ and $h^{-1} \circ \tau = \sigma \circ h^{-1}$.

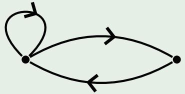
Overview

Motivating example: directed graphs

Let $E = (E^0, E^1, s, r)$ be a directed graph.

Example

The graphs



both determine $\mathcal{O}_2$. They have the same orbits (continuously orbit equivalent) but are not conjugate (Matsumoto).

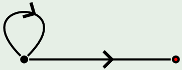
Motivating example: directed graphs

Remark

The boundary path space ∂E is locally compact, and the shift operation $\sigma_E \colon \partial E^{\geq 1} \to \partial E$ is a local homeomorphism.

Example

The graphs



have singular vertices.

Deaconu–Renault system

Definition

More generally, a Deaconu–Renault system is a locally compact Hausdorff space X equipped with a local homeomorphism $\sigma\colon \operatorname{dom}(\sigma) \to X$ between open subsets.

Example

- Discrete directed graphs; in particular, shifts of finite type;
- Nonexample: shifts of <u>infinite</u> type;
- Katsura's topological graphs;
- Integer actions on (locally) compact Hausdorff spaces.

C^*-algebras

A (partial) local homeomorphism $\sigma\colon X \to X$ on a locally compact Hausdorff space determines a C^*-algebra via a groupoid construction: $C^*(X,\sigma) = C^*(G_X)$ (Anantharaman-Delaroche, Deaconu, Renault).

Example

- Cuntz–Krieger algebra of shifts of finite type;
- Graph C^*-algebra of directed graph;
- Kirchberg algebras (via topological graphs);
- Crossed products by $\mathbb{Z}$.

Main question

Can we recover (X,σ) from $C^*(X,\sigma)$?

First step: diagonal subalgebra

The C^*-algebra $C^*(X, \sigma)$ carries a distinguished diagonal subalgebra $C_0(X)$.

Example

Simple Cuntz–Krieger algebras: diagonal is Cartan subalgebra (Renault).

Remark

The data $(C^*(X, \sigma), C_0(X))$ recovers the orbit structure of (X, σ). (Groupoid reconstruction: Renault, Carlsen–Ruiz–Sims–Tomforde).

Second step: gauge action

There is canonical gauge action $\gamma^X : \mathbb{T} \curvearrowright C^*(X, \sigma)$, and $C_0(X) \subset C^*(X, \sigma)^{\gamma^X}$.

Example

This is the canonical gauge action on graph C^*-algebras $\gamma^E : \mathbb{T} \curvearrowright C^*(E)$.

Remark

The data $(C^*(X, \sigma), C_0(X), \gamma^X)$ recovers (X, σ) up to eventual conjugacy (Matsumoto, Carlsen–Rout, CRST).

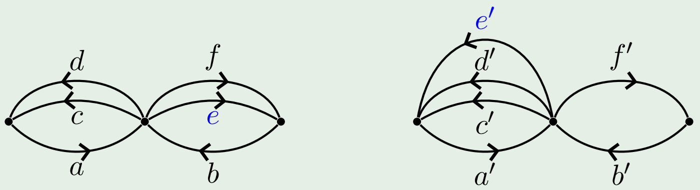
Second step: gauge action

Remark

The data $(C^*(X,\sigma), C_0(X), \gamma^X)$ recovers (X,σ) up to eventual conjugacy (Matsumoto, Carlsen–Rout, CRST).

Example (B–Carlsen)

The graphs



are eventually conjugate and not conjugate.

Third step: many gauge actions

Every continuous $f\colon X \to \mathbb{R}$ determines a weighted gauge action $\gamma^{X,f}\colon \mathbb{R} \curvearrowright C^*(X,\sigma)$.

Example

- An integer weight on the edges of a graph E produces a function $f\colon \partial E \to \mathbb{Z}$.
- Canonical gauge action comes from $f = 1$.

Question. Is the data $(C^*(X,\sigma), C_0(X), \{\gamma^{X,f}\}_{f\in C(X,\mathbb{R})})$ sufficient to recover (X,σ)?

Main result: YES

Theorem (Armstrong–B–Carlsen–Eilers)

Let (X, σ) and (Y, τ) be local homeomorphisms on locally compact Hausdorff spaces. They are conjugate if and only if there is a *-isomorphism $\varphi\colon C^*(X, \sigma) \to C^*(Y, \tau)$ satisfying

$$\varphi \circ \gamma_t^{X, g \circ h} = \gamma_t^{Y, g} \circ \varphi, \qquad g \in C(Y, \mathbb{R}), t \in \mathbb{R},$$

for some homeomorphism $h\colon X \to Y$.

Remark

Diagonal subalgebra $C_0(X)$ occurs as the joint fixed point algebra.

Main result for discrete graphs

Corollary

Let E and F be discrete directed graphs. They are conjugate if and only if there is a *-isomorphism $\varphi\colon C^*(E) \to C^*(F)$ satisfying

$$\varphi \circ \gamma_z^{E, g\circ h} = \gamma_z^{F,g} \circ \varphi, \qquad g \in C(\partial F, \mathbb{Z}), z \in \mathbb{T},$$

for some homeomorphism $h\colon \partial E \to \partial F$.

Question

Question

- Is the equivariant K-theory of the gauge action γ^f a useful invariant?
- Is there a similar result for two-sided conjugacy of shifts of finite type?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