The radius of comparison of the tensor product of a C*-algebra by $C(X)$

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- [AA20] With M. B. Asadi, The radius of comparison of the tensor product of a C*-algebra with $C(X)$, preprint (arXiv:2004.03013v1 [math.OA]). (To appear in the Journal of Operator Theory).

Warning: Some definitions are not stated carefully.

Introduction

The comparison theory of projections plays an important role in the type classification of factors.

Question: What if a C*-algebra has few or no projections?

In this case the comparison theory of projections can say little about the structure of the C*-algebra.

Question: What is an appropriate replacement for projections?

The appropriate substitute for projections is positive elements. This idea was first introduced by Cuntz with the purpose of studying dimension functions on simple C*-algebras. Later, the definition of the radius of comparison of C*-algebras, based on the Cuntz semigroup, was introduced by Andrew Toms to study exotic examples of simple amenable C*-algebras that are not $\mathcal{Z}$-stable.

Conjecture (Toms-Winter 2008)

If A is a simple stably finite separable amenable C*-algebra, then the following are equivalent:

1. A has finite nuclear dimension.
2. A is $\mathcal{Z}$-stable (where $\mathcal{Z}$ is the Jiang–Su algebra).
3. A has strict comparison of positive elements (i.e., $\mathrm{rc}(A) = 0$).

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Conjecture (Phillips-Toms 2015)

Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$.

For a compact metric space X, we use $\dim(X)$ to denote the covering dimension of X. For a compact manifold, the covering dimension agrees with the usual dimension of the manifold.

Conjecture (A.-Phillips 2019)

Let A be a stably finite unital C*-algebra and let X be a compact metric space. Then

$$\operatorname{rc}(A) \leq \operatorname{rc}\left(C(X) \otimes A\right) \leq \frac{1}{2}\dim(X) + \operatorname{rc}(A) + 1.$$

Definition

A C*-algebra is said to be strictly homogeneous if it is isomorphic to $p\big(C(X) \otimes \mathcal{K}\big)p$ for a compact Hausdorff space X and a projection of constant rank $p \in C(X) \otimes \mathcal{K}$. A strictly semihomogeneous C*-algebra is a finite direct sum of strictly homogeneous C*-algebras.
An approximately homogeneous (AH) C*-algebra is a direct limit

$$A = \varinjlim (A_j, \psi_j)$$

where A_j is strictly semihomogeneous for each j.

Definition

1. Let $A = \varinjlim (A_j, \psi_j)$ be a unital (i.e., both A_j and $\psi_j \colon A_j \to A_{j+1}$ are unital for every $j \in \mathbb{Z}_{>0}$) AH algebra, where

$$A_j = \bigoplus_{l=1}^{m_j} p_{j,l} \big(C(X_{j,l}) \otimes \mathcal{K} \big) p_{j,l}$$

   for compact Hausdorff spaces $X_{j,l}$, projections $p_{j,l} \in C(X_{j,l}) \otimes \mathcal{K}$, and natural numbers m_j. Define $\psi_{jk} = \psi_{j-1} \circ \psi_{j-2} \circ \cdots \circ \psi_k$, and write $\psi_{j\infty} \colon A_j \to A$ for the canonical map. We consider this collection of objects and maps as a decomposition for A. If ψ_j is injective for every j, then we describe this collection as an injective decomposition.

2. If a unital AH algebra A admits a decomposition as in (1) for which

$$\lim_{j \to \infty} \min_{1 \leq l \leq m_j} \big(\operatorname{rank}(p_{j,l}) \big) = \infty,$$

   then we say that A has large matrix sizes.

Definition (Toms 2006)

Let A be a unital AH algebra. The dimension-rank ratio of A, denoted $\mathrm{drr}(A)$, is the infimum of the set of strictly positive reals r such that A has a decomposition which satisfies

$$\limsup_{j\to\infty} \max_{1\leq l\leq m_j} \left(\frac{\dim(X_{j,l})}{\mathrm{rank}(p_{j,l})} \right) = r,$$

whenever this set is not empty, and ∞ otherwise.

Definition

Let A be a C*-algebra. Let $m, n \in \mathbb{Z}_{>0}$, let $a \in M_n(A)_+$, and let $b \in M_m(A)_+$.

1. We say that a is Cuntz subequivalent to b in A, written $a \precsim_A b$, if there exists a sequence $(x_k)_{k=1}^{\infty}$ in $M_{n,m}(A)$ such that

$$\lim_{k \to \infty} x_k b x_k^* = a.$$

2. We say that a is Cuntz equivalent to b in A, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$.

Example

Let X be a compact metric space and let $f, g \in C(X)_+$. Then

$$f \precsim_{C(X)} g \Longleftrightarrow \{x \in X \colon f(x) \neq 0\} \subseteq \{x \in X \colon g(x) \neq 0\}.$$

Example

Let $n \in \mathbb{Z}_{>0}$, let $A = M_n\big(C([0,1])\big)$, and let $f, g \in A_+$. Then

$$f \precsim_A g \Longleftrightarrow \operatorname{rank}(f(t)) \leq \operatorname{rank}(g(t)) \quad \text{for all } t \in [0,1].$$

We let $T(A)$ be the set of all normalized traces on A, (without renormalization: the trace of the identity of $M_n(A)$ is n, not 1).

Notation

Let A be a stably finite exact unital C*-algebra. For every $\tau \in \mathrm{T}(A)$ and every $a \in \bigcup_{k=1}^{\infty} M_k(A)_+$, set

$$d_\tau(a) = \lim_{n \to \infty} \tau(a^{\frac{1}{n}}).$$

Definition

Let A be a stably finite exact unital C*-algebra.

1. For $r \in [0, \infty)$, A has r-comparison if whenever $a, b \in \bigcup_{k=1}^{\infty} M_k(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $a \precsim_A b$.
2. The radius of comparison of A, denoted $\mathrm{rc}(A)$, is
$$\mathrm{rc}(A) = \inf\left(\left\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\right\}\right).$$
(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

If $A = C(X)$, the condition $d_\tau(a) + r < d_\tau(b)$ for all τ becomes
$$\int_X \mathrm{rank}(a(x))\, d\mu(x) + r < \int_X \mathrm{rank}(b(x))\, d\mu(x)$$
for every Borel probability measure μ on X which is the same as
$$\mathrm{rank}(a(x)) + r < \mathrm{rank}(b(x))$$
for all x in X.

Proposition (Toms 2006)

Let A and B be stably finite unital C*-algebras. Then:

1. $\mathrm{rc}(A \oplus B) = \max\big(\mathrm{rc}(A), \mathrm{rc}(B)\big)$.
2. If $k \in \mathbb{Z}_{>0}$, then $\mathrm{rc}\big(M_k(A)\big) = \frac{1}{k} \cdot \mathrm{rc}(A)$.

Theorem (rc of a corner)

Let A be a stably finite exact unital C*-algebra and let p be a full projection in A. Define

$$\lambda = \inf\big(\{\tau(p) \colon \tau \in \mathrm{T}(A)\}\big) \qquad \text{and} \qquad \eta = \sup\big(\{\tau(p) \colon \tau \in \mathrm{T}(A)\}\big).$$

Then $0 < \lambda \leq \eta \leq 1$ and

$$\frac{1}{\eta} \cdot \mathrm{rc}(A) \leq \mathrm{rc}(pAp) \leq \frac{1}{\lambda} \cdot \mathrm{rc}(A).$$

Many results about the Cuntz semigroup of $C(X)$ when $\dim(X) \leq 3$ can be found in the work of Robert and Tikuisis.

Theorem (Elliott-Niu 2013)

Let X be a finite CW-complex.

1. If $\dim(X)$ is even, then

$$\frac{\dim(X)}{2} - 2 \leq \mathrm{rc}\big(C(X)\big) \leq \max\left(0, \frac{\dim(X)}{2} - 1\right).$$

2. If $\dim(X)$ is odd, then

$$\mathrm{rc}\big(C(X)\big) = \max\left(0, \frac{\dim(X) - 1}{2} - 1\right).$$

Proposition

The right-hand side of the conjecture on $\mathrm{rc}(C(X) \otimes A)$ is true for the case $A = M_n(C(Y))$ for $n \in \mathbb{Z}_{>0}$ with technical hypotheses on X and Y.

Remark (Toms 2006)

For every compact Hausdorff space X and projection $p \in C(X) \otimes \mathcal{K}$, we have

1. $\mathrm{rc}\Big(p(C(X) \otimes \mathcal{K})p\Big) \leq \max\left(0, \frac{\dim(X)-1}{2\,\mathrm{rank}(p)}\right)$.
2. $\mathrm{drr}\Big(p\big(C(X) \otimes \mathcal{K}\big)p\Big) = \frac{\dim(X)}{\mathrm{rank}(p)}$.

Lemma (Mostly based on Toms 2006)

Let A be a unital AH algebra. Then $\mathrm{rc}(A) \leq \frac{1}{2}\mathrm{drr}(A)$.

Proposition (Mostly based on Toms 2006)

For every $r \in [0, \infty)$, there exists a simple unital AH algebra such that $\mathrm{rc}(A) = \frac{1}{2}\mathrm{drr}(A) = r$.

Theorem

Let X be a compact metric space and let A be a stably finite unital C*-algebra. Then $\mathrm{rc}(A) \leq \mathrm{rc}\big(C(X) \otimes A\big)$.

There are some difficulties to prove the conjecture on $\mathrm{rc}(C(X) \otimes A)$ for a general C*-algebra A and a metric space X. However, we could show that it is valid at least for some choices of A and X.

Proposition

Let X be a compact metric space with $m = \dim(X) < \infty$. Let A be a unital C*-algebra, let $l \in \mathbb{Z}_{>0}$, and let $a, b \in M_l(C(X, A))_+$. If $a(x) \precsim_A b(x)$ for all $x \in X$, then $a \precsim_{C(X,A)} 1_{M_{m+1}} \otimes b$.

Proposition

The right-hand side of the conjecture on $\mathrm{rc}(C(X) \otimes A)$ is true for the case $\dim(X) = 0$.

Theorem

Let A be a residually stably finite $\mathcal{Z}$-stable unital C*-algebra. Then $\mathrm{rc}(C(X) \otimes A) = 0$.

To prove the above theorem, we apply the results of [BRTTW12]. Therefore, finiteness and residual stable finiteness of $C(X, A)$ is one of the starting points.

It is a result of Rørdam that $\mathrm{rc}(A) = 0$ for a stably finite exact simple $\mathcal{Z}$-stable unital C*-algebra A. To see that not all $\mathcal{Z}$-stable unital C*-algebras have strict comparison of positive elements, we exhibit a class of stably finite exact $\mathcal{Z}$-stable unital C*-algebras with nonzero radius of comparison.

Definition

A C*-algebra A is said to be purely infinite if, for every nonzero positive element $a \in A$, we have $a \oplus a \precsim_A a$.

Definition

Let A be a C*-algebra. The unitization of a C*-algebra A is denoted by A^+. The cone over A, denoted CA, is the set of continuous functions $f\colon [0,1] \to A$ with $f(0) = 0$.

Proposition

Let A be a purely infinite exact simple unital C*-algebra and let B be a stably finite exact unital C*-algebra. Then

$$\mathrm{rc}\big((CA)^+\big) = \mathrm{rc}\big((CA)^+ \otimes_{\min} B\big) = \infty.$$

If the C*-algebra B in the above proposition is also $\mathcal{Z}$-stable, then we can get a class of stably finite exact $\mathcal{Z}$-stable unital C*-algebras with nonzero radius of comparison. For example, take $A = \mathcal{O}_2$ and $B = \mathcal{Z}$.

Theorem

Let A be a unital AH algebra with large matrix sizes and let X be a compact metric space. Then the dimension-rank ratios of A and $C(X) \otimes A$ are related by

$$\mathrm{drr}(A) = \mathrm{drr}\big(C(X) \otimes A\big).$$

Theorem

Let A be a unital AH algebra with large matrix sizes and let X be a compact metric space. Suppose $\mathrm{rc}(A) = \frac{1}{2}\mathrm{drr}(A)$. Then:

1. $\mathrm{rc}(C(X) \otimes A) = \frac{1}{2}\mathrm{drr}(C(X) \otimes A)$.
2. $\mathrm{rc}(C(X) \otimes A) = \mathrm{rc}(A)$.

Example ([A.-Golestani-Phillips 2019])

Now we give an example of a simple AH algebra A with $\mathrm{rc}(A) > 0$ and an action $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$ which has the Rokhlin property. We use two copies of the same system. Writing the direct system sideways, our combined system looks like the following diagram, in which the solid arrows represent many partial maps and the dotted arrows represent a small number of point evaluations:

$$
\begin{tikzcd}
C(X_1) \otimes M_{r(1)} \arrow[r, Rrightarrow] \arrow[dr, dotted] & C(X_2) \otimes M_{r(2)} \arrow[r, Rrightarrow] \arrow[dr, dotted] & C(X_3) \otimes M_{r(3)} \arrow[r, Rrightarrow] \arrow[dr, dotted] & \cdots \\
C(X_1) \otimes M_{r(1)} \arrow[r, Rrightarrow] \arrow[ur, dotted] & C(X_2) \otimes M_{r(2)} \arrow[r, Rrightarrow] \arrow[ur, dotted] & C(X_3) \otimes M_{r(3)} \arrow[r, Rrightarrow] \arrow[ur, dotted] & \cdots .
\end{tikzcd}
$$

The above construction is motivated by the work of I. Hirshberg and N. C. Phillips.

Example

Let X be a compact metric space and let $A = \varinjlim A_n$ be the AH algebra in the previous example. Then:

1. $\mathrm{rc}(A) = \frac{1}{2}\mathrm{drr}(A)$.
2. $\mathrm{rc}(C(X) \otimes A) = \frac{1}{2}\mathrm{drr}(C(X) \otimes A)$.
3. $\mathrm{rc}(C(X) \otimes A) = \mathrm{rc}(A)$.

Theorem

For every $r \in [0, \infty)$ and for every compact metric space X, there exists a simple unital AH algebra such that $\mathrm{rc}(C(X) \otimes A) = \mathrm{rc}(A) = r$.

Thank you for your attention!