OPERATOR THEORY AND COARSE GEOMETRY

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At Scottish Operator Algebras Seminar

28 Nov, 2014

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Outline

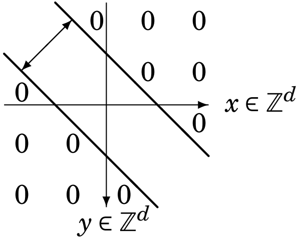
1. Limit Operator Theory
2. Coarse Geometry and Roe C*-algebras
3. Uniform boundedness of inverses
4. Limit operators in general

Band Operators

Think of $\mathbb{Z}^d$ as a metric space, with "absolute value" metric: $d((x_i),(y_i)) = \sum |x_i - y_i|$.

Consider operators $T \in \mathscr{B}(\ell^2\mathbb{Z}^d)$ as matrices $T = (T_{xy})_{x,y\in\mathbb{Z}^d}$.

Such a T is called a **band operator**, if there exists $R \geq 0$, such that $T_{xy} = 0$ whenever $d(x,y) > R$. [Also called *finite propagation operators*.]



Band–dominated operators = norm–limits of band operators.

Can run this in $\ell^2(\mathbb{Z}^d, E)$, where E is a Banach space [then the matrix entries are ops in $\mathscr{B}(E)$]; Also $\ell^p(\cdots)$, $p \in [1,\infty]$.

Note: These notions make sense over any countable metric space X with bounded geometry in place of $\mathbb{Z}^d$!
[bdd.geom. = number of points in balls of fixed radius is unif. bounded]

Limit Operators [Rabinovich–Roch–Silbermann, '80s – now]

Let $U_g \in \mathscr{B}(\ell^2\mathbb{Z}^d)$ be the unitary associated to $g \in \mathbb{Z}^d$, the "shift by g": defined as $U_g\xi(h) = \xi(-g+h)$ for $\xi \in \ell^2\mathbb{Z}^d$, $h \in \mathbb{Z}^d$; or equivalently $U_g\delta_h = \delta_{g+h}$, for $g,h \in \mathbb{Z}^d$; where $\{\delta_g \mid g \in \mathbb{Z}^d\}$ is the usual basis of $\ell^2\mathbb{Z}^d$ consisting of "point mass" functions.

Fix an operator $T \in \mathscr{B}(\ell^2\mathbb{Z}^d)$. Given a sequence $(g_n) \subset \mathbb{Z}^d$ converging to ∞, consider the sequence of shifts of T: $(U_{g_n^{-1}}TU_{g_n})_{n\in\mathbb{N}}$.

If it has a *-strongly convergent subsequence, we call the limit $T_{(g_n)} \in \mathscr{B}(\ell^2\mathbb{Z}^d)$ a *limit operator* of T associated to (g_n).
The set of all limit operators of T is called the *operator spectrum*, $\sigma_{\mathrm{op}}(T)$.

The collection $\mathscr{A}_{R,N} \subset \mathscr{B}(\ell^2\mathbb{Z}^d)$ of operators with band-width at most R and norm at most N is *-strongly compact. Thus any band–dominated operator has a limit operator associated with any sequence (g_n).

Note: This construction works on any countable group Γ in place of $\mathbb{Z}^d$.

Limit Operators: An example

Slowly oscillating coefficients

A function $f \in \ell^\infty(\mathbb{Z}^d)$ acts on $\ell^2(\mathbb{Z}^d)$ by multiplication (diagonal operator). $\sigma_{\text{op}}(f)$ can be very complicated.

An $f \in \ell^\infty(\mathbb{Z}^d)$ is *slowly oscillating*, if for all $r, \varepsilon > 0$ there exists a finite $F \subset \mathbb{Z}^d$, such that

$$\sup_{x \in \mathbb{Z}^d \setminus F} \left(\sup_{d(x,y) \leq r} |f(x) - f(y)| \right) \leq r.$$

Then all limit operators of f are scalars, and

$$\sigma_{\text{op}}(f) = \bigcap_{\text{finite } F \subset \mathbb{Z}^d} \overline{f(\mathbb{Z}^d \setminus F)} \subseteq \mathbb{C}.$$

Elaborating, the band operators of the form $T = \sum f_g U_g$ with slowly oscillating f_g's have their limit operators in the group ring $\mathbb{C}[\mathbb{Z}^d]$; the band–dominated ones in $C^*(\mathbb{Z}^d) \cong C(\mathbb{T}^d)$.

Fredholmness criterion

Theorem (Rabinovich–Roch–Silbermann, '90s)

A band-dominated operator on $\ell^p(\mathbb{Z}^d)$ is Fredholm iff all its limit operators are invertible, with uniform bound on the norm of the inverses.

Remark: Can do also on $\ell^p(\mathbb{Z}^d, E)$ for a Banach space E, with appropriately adjusted notion of "Fredholm".

Question ("Core Issue for Limit Operators", '90s)

Can we drop the "uniform bound on the norm on the inverses" requirement? I.e., is the operator spectrum of a band–dominated operator automatically uniformly invertible as soon as it is pointwise invertible?

History: answered positively for various classes of band–dominated operators: e.g. "ℓ^1-type" (Wiener); with slowly oscillating coefficients.

Theorem (Lindner–Seidel, '14)

Yes, for any band–dominated operator on $\ell^p(\mathbb{Z}^d, E)$.

Main Theorem

Theorem (S–Willett)

Let X be a countable metric space with bounded geometry, $p \in (1,\infty)$. Assume that X has Yu's property A. Let T be a band-dominated operator on $\ell^p(X)$. Then the following are equivalent:

- *T is Fredholm.*
- *All limit operators of T are invertible, and the norms of their inverses are uniformly bounded.*
- *All limit operators of T are invertible.*

- What's Yu's property A and why do we need it?
- What are limit operators in the general setting?

Remark: Can do also on $\ell^p(X,E)$ for a Banach space E, with the "usual" modifications to Fredholmness.

Remark: Fails if X does not have property A.

METRIC SPACES AND C*-ALGEBRAS

Let X be a metric space. For the rest of the talk, assume that X is countable, discrete, and has *bounded geometry*, i.e. there's a uniform bound on the cardinality of balls of fixed radius.

Examples: Vertex sets of graphs (e.g. countable groups); "discretizations" of (non-compact) Riemannian manifolds.

Example: If Γ is a group generated by a finite set $S \subset \Gamma$, then

$$d_S(g,h) = \min\{n \in \mathbb{N} \mid g^{-1}h = s_1^{\pm 1} \cdots s_n^{\pm 1}, s_1, \dots, s_n \in S\}$$

is a (left-invariant) metric on Γ [usually called a "word metric"].

Example: $\mathbb{Z}^d$ is generated by $S = \{(1,0,\dots,0),\dots,(0,\dots,0,1)\}$. Then d_S is the "absolute value" metric we've seen before.

Defn: The C*-algebra $\mathscr{A}$ of band-dominated operators on $\ell^2(X)$ is called the *uniform Roe algebra* of X.

Defn: A fn $f : X \to Y$ is a *coarse equivalence*, if $\exists \rho_-, \rho_+ : \mathbb{R}^+ \to \mathbb{R}^+$, $\rho_- \nearrow \infty$, such that $\forall x,y \in X$: $\rho_-(d_X(x,y)) \leq d_Y(f(x),f(y)) \leq \rho_+(d_X(x,y))$.

LIMIT OPS FOR GROUPS REVISITED (BY J. ROE '04) I.

Work on a countable group Γ (a metric space). Denote by $U_g \in \mathscr{B}(\ell^2\Gamma)$ the unitary given on the basis $\{\delta_g \mid g \in \Gamma\} \subset \ell^2\Gamma$ as $U_g\delta_h = \delta_{gh}$.

Sequences $(g_n) \subset \Gamma$ tending to ∞ $\leftrightsquigarrow$ points in $\partial\Gamma = \beta\Gamma \setminus \Gamma$, the Stone-Čech boundary of Γ.

Use $\partial\Gamma$ for indexing limit ops; construct them all at once:

Construction: Fix $T \in \mathscr{A}$. Consider the map $\Gamma \to \mathscr{A}$

$$g \mapsto U_{g^{-1}} T U_g$$

($\mathscr{A}$ with *-strong topology). Extend to a *-str continuous map ("symbol")

$$\sigma_{\mathrm{op}}(T) : \partial\Gamma \to \beta\Gamma \to \mathscr{A}.$$

Denote $C_s(\partial\Gamma;\mathscr{A})$ the C*-algebra of *-str ctns maps; $\sigma_{\mathrm{op}}(T) \in C_s(\partial\Gamma;\mathscr{A})$.

Point: $\sigma_{\mathrm{op}} : \mathscr{A} \to C_s(\partial\Gamma;\mathscr{A})$ is a *-homomorphism.

The kernel of σ_{op} consists of *ghost operators*.

Defn (G. Yu): $T \in \mathscr{A}$ is a *ghost*, if its matrix entries tend to 0 at infinity.
NB: Compact operators are always ghosts.

Limit ops for groups revisited (by J. Roe '04) II.

Recall: $\{\text{ghosts}\} = \ker(\sigma_{\mathrm{op}}) \subset \mathscr{A} \xrightarrow{\sigma_{\mathrm{op}}} C_s(\partial\Gamma; \mathscr{A})$

Definition

A group Γ is has *Yu's property A* [$\Longleftrightarrow$ is *exact*], if taking minimal crossed product of terms of any short exact sequence of Γ-C*-algebras by Γ preserves its exactness [$\Longleftrightarrow$ $C^*_{\mathrm{red}}\Gamma$ is an exact C*-algebra].

Theorem ($\Leftarrow$ Yu '00, $\Rightarrow$ Roe–Willett '13)

All ghosts in $\mathscr{A}$ are compact iff Γ has property A.

Theorem (Roe '04)

Let Γ have property A, $T \in \mathscr{A} \subset \mathscr{B}(\ell^2\Gamma)$. TFAE

(i) *T is Fredholm,*

(ii) *$\sigma_{\mathrm{op}}(T)$ is invertible in $C_s(\partial\Gamma; \mathscr{A})$,*

(iii) *Each $S \in \sigma_{\mathrm{op}}(T)$ is invertible and $\sup_{S\in\sigma_{\mathrm{op}}(T)} \|S^{-1}\| < \infty$.*

More about Property A / exactness

Why?

It's a "niceness" / "regularity" condition. Many equivalent formulations (coarse geometric, analytic (approximation properties), dynamical (actions on compact spaces)).

Strong consequences: e.g. Novikov conjecture in topology (via K-theory of Roe C*-algebras / index theory).

Examples/Groups:

Groups with A: amenable, linear, hyperbolic, mapping class groups. Also "finite-dimensional" groups (e.g. $\mathbb{Z}^d$).

Groups without A: Gromov Monsters: "groups containing non-property-A families of graphs in their Cayley graph". [Hard to construct.]

Unknown: $Out(\mathbb{F}_n)$, Thompson's group F.

Examples/Spaces:

Without A: graphs of large girth, expanders.

Lindner–Seidel's proof of "Core Issue"

Proposition

On $\ell^p(\mathbb{Z}^d)$: If $T \in \mathscr{A}$ and all $S \in \sigma_{\mathrm{op}}(T)$ are invertible, then $\exists S \in \sigma_{\mathrm{op}}(T)$ with $\|S^{-1}\| = \sup_{B \in \sigma_{\mathrm{op}}(T)} \|B^{-1}\|$.

Defn: The *lower norm* $\nu(T)$ of any operator $T \in \mathscr{B}(E)$ is defined to be $\nu(T) = \inf\left\{\frac{\|T\psi\|}{\|\psi\|} \mid \psi \in E \setminus \{0\}\right\}$. [So $\nu(T) = 1/\|T^{-1}\|$ for an invertible T.]
If $E = \ell^2\Gamma$, we can talk about *support* of a vector $\psi \in \ell^2\Gamma$ as a subset of Γ.
Then the *localised lower norm* (for $D \geq 0$) of $T \in \mathscr{B}(\ell^2\Gamma)$ is

$$\nu_D(T) = \inf\left\{\frac{\|T\psi\|}{\|\psi\|} \mid \psi \in \ell^2\Gamma \setminus \{0\}, \mathrm{diam}(\mathrm{supp}(\psi)) \leq D\right\}.$$

Lemma (Localisation; Step 1 of L-S proof)

On $\ell^p(\mathbb{Z}^d)$: Given $\delta > 0$, $R \geq 0$, $N \geq 0$, there exists $D \geq 0$, such that for all $T \in \mathscr{A}_{R,N}$ [band-width $\leq R$, norm $\leq N$] we have $\nu_D(T) \leq \nu(T) + \delta$.

Step 2: An "accumulation of singularities" argument; doesn't actually use $\mathbb{Z}^d$ – valid for any group.

METRIC SPARSIFICATION PROPERTY

DEFINITION (CHEN–TESSERA–WANG–YU '07)

A metric space X has the Metric Sparsification Property with $c \in (0,1]$, if $\exists$ non-decreasing $f : \mathbb{N} \to \mathbb{N}$, such that:
For any $m \in \mathbb{N}$ and any finite positive Borel measure μ on X there exists $\Omega = \sqcup_{i \in I} \Omega_i \subset X$ with

- $d(\Omega_i, \Omega_j) \geq m$ whenever $i \neq j \in I$;
- $\operatorname{diam}(\Omega_i) \leq f(m)$ for every $i \in I$;
- $\mu(\Omega) \geq c\mu(X)$.

Remark: MSP $\iff$ Yu's property A [Brodzki–Niblo–S–Willett–Wright; Sako '12]

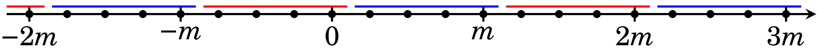
PROPOSITION (S–WILLETT)

X has MSP $\implies$ Localisation Lemma of L–S on $\ell^p(X)$.

Idea: Use MSP to chop T into a block–diagonal shape.

Metric Sparsification for $\mathbb{Z}$

Let $f(m) = m$, $c = \frac{1}{2}$. Given $m \in \mathbb{N}$, consider



Given any finite measure μ on $\mathbb{Z}$ (assignment of weights to points of $\mathbb{Z}$), either the red "half" of $\mathbb{Z}$, or the blue "half" of $\mathbb{Z}$, has measure $\geq \frac{1}{2}$. Denote that "half" by Ω. So $\mu(\Omega) \geq \frac{1}{2}\mu(X)$.

Whichever Ω is, it naturally splits into m-separated intervals, each of diameter $m = f(m)$.

Limit spaces

If Γ is a group, we used "shifts" to construct limit ops (all on $\ell^p\Gamma$). Don't have these in general.

Let X be a countable, bounded geometry metric space.
Let $\omega \in \partial X = \beta X \setminus X$, i.e. a non-principal ultrafilter on X.

We associate a (ctbl, bdd.geom.) metric space to ω: $X(\omega)$, the *limit space of X at ω*. Comes with a distinguished point, $\omega \in X(\omega)$.

Proposition: For any $R \geq 0$, there exists $Y \subset X$, such that

- $Y \in \omega$ and
- $\forall y \in Y$, the R-ball $B_R(y) \subset X$ is isometric to $B_R(\omega) \subset X(\omega)$.

Example: If $X = \Gamma$ is a countable group, then $X(\omega) \cong \Gamma$ for all $\omega \in \partial\Gamma$.

Example: If $X = \mathbb{N}$ with the natural metric, then all limit spaces are $\cong \mathbb{Z}$.

Example: If X consists of "spheres of radius n^2", $n \in \mathbb{N}$ inside $\mathbb{Z} \times \mathbb{Z}$ with ℓ^∞-metric, then all limit spaces are $\cong \mathbb{Z}$.

Limit operators

Limit operators (of an op on $\ell^2(X)$) will "live" on $\ell^2(X(\omega))$, $\omega \in \partial X$.

Notation: For $S \subset X$, denote $P_S \in \mathscr{B}(\ell^2(X))$ the orthogonal projection onto $\ell^2(S) \subset \ell^2(X)$.

"Construction": Take a band op T on $\ell^2(X)$ and $\omega \in \partial X$.

Given $R \geq 0$, by the Proposition, there is $Y_R \subset X$ with $Y_R \in \omega$, such that all balls $B_R(y)$, $y \in Y$, are isometric [to $B_R(\omega)$]. The "cut-downs" $P_{B_R(y)} T P_{B_R(y)}$ are finite matrices of the same "shape".

So, by passing to a subset of Y_R, we can arrange that the "cut-down" matrices entry-wise converge to a single matrix, which we declare to be $P_{B_R(\omega)} \Phi_\omega(T) P_{B_R(\omega)} \in \mathscr{B}(\ell^2(X(\omega)))$, the "cut-down" of (so far non-existent) limit operator $\Phi_\omega(T)$.

Now, increase R and repeat. The actual $\Phi_\omega(T)$ will then be strong operator limit of the "matrices" on $\ell^2(X(\omega))$ that we constructed.