On ℓ-open and ℓ-closed C^*-algebras

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Abstract. In this paper, we characterize ℓ-open and ℓ-closed C^*-algebras and deduce that ℓ-open C^*-algebras are ℓ-closed, as conjectured by Blackadar. Moreover, we show that a commutative unital C^*-algebra is ℓ-open if and only if it is semiprojective.

1. Introduction

Lifting properties of C^*-algebras and their *-homomorphisms have been well-studied for some time with prominent connections to notions of stability; see [1, 10, 14, 15] for example. They play an important role in modern C^*-algebra theory including the Elliott classification program ([12, 9, 17], for example). In connection to a non-commutative generalization of Borsuk's homotopy extension theorem, Blackadar [3] defined natural classes C^*-algebras in terms of lifting properties, called ℓ-open and ℓ-closed C^*-algebras. A C^*-algebras is ℓ-open if the liftable maps from the C^*-algebra to any quotient C^*-algebra is a point-norm open set, and ℓ-closedness is defined similarly (precise definitions can be found in Section 2).

While these notions are first formalized only recently by Blackadar, their study traces back at least to the celebrated work of Brown, Douglas, and Fillmore: in [5], they seek conditions on a space X that ensure the set of liftable maps from $C(X)$ to the Calkin algebra is closed. It is open whether $C(\mathbb{D})$ is ℓ-closed, and a positive answer would settle an open question on page 119 of [4]. More recently, Enders and Shulman further studied when the set of liftable maps from $C(X)$ to the Calkin algebra is closed, including a sufficient condition when $\dim(X) \leq 2$ and a full characterization when $\dim(X) \leq 1$ [11].

In this paper, we prove the following characterizations of being ℓ-open and ℓ-closed:

Theorem 1.1 (see Theorem 3.8). *Let A be a C^*-algebra. The following are equivalent:*

(i) *A is ℓ-open.*

(ii) *For every C^*-algebra B and ideal $I \subseteq B$, the natural map* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ *is open.*
(iii) *A satisfies the Homotopy Lifting Theorem (a noncommutative analog of the Borsuk Homotopy Extension Theorem), and* $\mathrm{Hom}(A, B)$ *is locally path-connected for every C^*-algebra B.*

Condition (ii) can be strengthened to uniform openness (see Theorem 3.8).

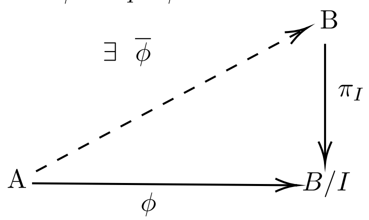
Theorem 1.2 (see Theorem 4.1**).** *Let A be a separable C^*-algebra. Then A is ℓ-closed if and only if for every C^*-algebra B and ideal $I \subseteq B$, the natural map* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ *is uniformly relatively open.*

As a consequence, we confirm a conjecture of Blackadar from [3], that ℓ-open C^*-algebras are ℓ-closed.

Additionally, we prove that a unital commutative C^*-algebra is semiprojective if and only if it is ℓ-open, confirming another conjecture from [3, Page 299].

2. Preliminaries

Let A and B be C^*-algebras and let I an ideal in B (by which we mean a closed, two-sided ideal). We write $\pi_I : B \to B/I$ be the quotient map. Recall that a *-homomorphism $\phi : A \to B/I$ is *liftable* if there exists a *-homomorphism $\overline{\phi} : A \to B$ such that $\phi = \pi_I \circ \overline{\phi}$:



We denote the space of *-homomorphisms from A to B endowed with the point-norm topology by $\mathrm{Hom}(A, B)$ and the subspace of unital *-homomorphisms by $\mathrm{Hom}_1(A, B)$ (if A and B are unital). For $\phi \in \mathrm{Hom}(A, B)$, a neighbourhood base of ϕ is made up of sets

$$U_B(\phi; \mathcal{F}, \epsilon) := \{\psi \in \mathrm{Hom}(A, B) : \|\psi(a) - \phi(a)\| < \epsilon \ \forall a \in \mathcal{F}\}, \tag{2.1}$$

ranging over all finite sets $\mathcal{F} \subset A$ and all positive real numbers $\epsilon > 0$. This gives a uniform structure to $\mathrm{Hom}(A, B)$. In fact, the sets of this neighbourhood base are parametrized independently of B, giving a uniform structure to all of $\mathrm{Hom}(A, B)$ at once. (One would like to put a uniform structure on the disjoint union of $\mathrm{Hom}(A, B)$ ranging over all C^*-algebras B, except that this is not a well-founded set. One can put a uniform structure on $\coprod_{B \in \mathcal{B}} \mathrm{Hom}(A, B)$, for any set $\mathcal{B}$ of C^*-algebras.)

The set of liftable *-homomorphisms $A \to B/I$ is

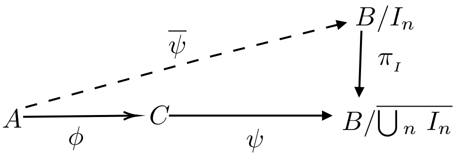
$$\mathrm{Hom}(A, B, I) := \pi_I \circ \mathrm{Hom}(A, B). \tag{2.2}$$

The following is due to Blackadar [3, Definition 6.1].

Definition 2.1. Let A be a C^*-algebra

(i) A is *ℓ-open* if for C^*-algebra B and every ideal I of B, the set $\mathrm{Hom}(A, B, I)$ is open in $\mathrm{Hom}(A, B/I)$.
(ii) A is *ℓ-closed* if for C^*-algebra B and every ideal I of B, the set $\mathrm{Hom}(A, B, I)$ is closed in $\mathrm{Hom}(A, B/I)$.

Definition 2.2. Recall that a *-homomorphism $\phi : A \to C$ is *(weakly) semiprojective* if for any C^*-algebra B, any increasing sequence $I_1 \lhd I_2 \lhd \cdots \lhd B$ of ideals in B, and any *-homomorphism $\psi : C \to B/\overline{\bigcup_n I_n}$ (and finite set $F \subset A$, $\epsilon > 0$), there is an n and a *-homomorphism $\overline{\psi} : A \to B/I_n$ such that $\psi \circ \phi = \pi_I \circ \overline{\psi}$ (resp. $\|\psi \circ \phi(x) - \pi_I \circ \overline{\psi}(x)\| < \epsilon$ for all $x \in F$), where $\pi_I : B/I_n \to B/I$ is the quotient map.



A is *(weakly) semiprojective* if the identity *-homomorphism is (weakly) semiprojective. Some examples of semiprojective C^*-algebras are finite dimensional C^*-algebras, the universal C^*-algebras generated by n unitaries, $C^*(\mathbb{F}_n)$, and $\{f \in C(S^1, \mathbf{M}_n) : f(1) \text{ is scalar}\}$ (see [15]).

Example 2.3. [3, Corollary 6.2] All semiprojective C^*-algebras are both ℓ-open and ℓ-closed C^*-algebras.

By slight abuse of notation, if $L \subseteq K \subseteq B$ are ideals, then we also use π_K to denote the quotient map from B/L to B/K.

We recall the following general Chinese remainder theorem for C^*-algebras:

Lemma 2.4 ([3], Proposition 2.1). *Let B be a C^*-algebra, and I and J ideals in B. Then $B/(I \cap J)$ is isomorphic to the fibred product $\{(x, y) \in x \in B/I \oplus y \in B/J : \pi_{I+J}(x) = \pi_{I+J}(y)\}$ via the map $a \to (\pi_I(a), \pi_J(a))$.*

3. Properties and characterization of ℓ-open C^*-algebras

The following shows that if A is ℓ-open then the quotient map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ is always open. In fact, it shows that this openness is uniform, as the relationship between $(\mathcal{G}, \delta)$ and $(\mathcal{F}, \epsilon)$ in the statement below does not depend on the C^*-algebra B, the ideal I, nor any of the *-homomorphisms under consideration. The conclusion of the following theorem is (in the separable case) a reformulation of the conclusion of [3, Theorem 4.1]; the ideas in the proof are similar, but work is needed to allow ℓ-openness instead of semiprojectivity as the hypothesis.

Theorem 3.1. *Let A be an ℓ-open C^*-algebra. Then for any $\epsilon > 0$ and any finite set $\mathcal{F} \subset A$, there is a $\delta > 0$ and a finite set $\mathcal{G} \subset A$ such that whenever B is a C^*-algebra, I is an ideal of B, γ and φ are *-homomorphisms from A to B/I with $\|\gamma(u) - \varphi(u)\| < \delta$ for all $u \in \mathcal{G}$ and such that γ lifts to a *-homomorphism $\overline{\gamma} : A \to B$, then φ also lifts to a *-homomorphism $\overline{\varphi} : A \to B$ with $\|\overline{\gamma}(v) - \overline{\varphi}(v)\| < \epsilon$ for all $v \in \mathcal{F}$. In other words, in the notation of* (2.1),

$$U_{B/I}(\gamma; \mathcal{G}, \delta) \subseteq \pi_I \circ U_B(\overline{\gamma}; \mathcal{F}, \epsilon). \tag{3.1}$$

Proof. Let $(\mathcal{G}_n)_{n\in\Lambda}$ be an increasing net of finite subsets of A whose union is dense in A, and let $(\delta_n)_{n\in\Lambda}$ be a net (over the same index set) of positive numbers such that $\delta_n \to 0$. Suppose that the conclusion of the theorem is false for a fixed $\epsilon > 0$ and finite set $\mathcal{F}$. Then, there are C^*-algebras B_n with ideals I_n and *-homomorphisms $\gamma_n, \varphi_n : A \to B_n/I_n$ such that

$$\|\gamma_n(u) - \varphi_n(u)\| < \delta_n \tag{3.2}$$

for all $u \in \mathcal{G}_n$, γ_n lifts to $\overline{\gamma}_n : A \to B_n$, but no φ_n lifts to *-homomorphism $\overline{\varphi}_n : A \to B_n$ with $\|\overline{\gamma}_n(v) - \overline{\varphi}_n(v)\| < \epsilon$ for all $v \in \mathcal{F}$.

Let $B := \prod_{n\in\Lambda} B_n$, $I := \prod_{n\in\Lambda} I_n$, and $J := \{(b_n) \in B : \lim_n \|b_n\| = 0\}$. Then $B/I \cong \prod_{n\in\Lambda} B_n/I_n$. Define *-homomorphisms $\overline{\gamma} := (\overline{\gamma}_n)_{n\in\Lambda} : A \to B$ and $\varphi := (\varphi_n)_{n\in\Lambda} : A \to B/I$. Then (3.2) implies that $\lim_n \|\gamma_n(x) - \varphi_n(x)\| = 0$ for all $x \in A$, and so $\pi_{I+J} \circ \overline{\gamma} = \pi_{I+J} \circ \varphi$.

Using the general Chinese remainder theorem (Lemma 2.4), there exists a *-homomorphism $\theta : A \to B/(I \cap J)$ such that

$$\pi_J \circ \overline{\gamma} = \pi_J \circ \theta \quad \text{and } \varphi = \pi_I \circ \theta \tag{3.3}$$

Take a *-linear lift $(\theta_n)_{n\in\Lambda} : A \to B$ of θ (which need not be a *-homomorphism), thus defining $\theta_n : A \to B_n$. For $m \in \Lambda$, define $\alpha_m := \pi_{I\cap J} \circ (\alpha_{m,n})_{n\in\Lambda}$, where

$$\alpha_{m,n} := \begin{cases} \theta_n, & n \geq m; \\ \overline{\gamma}_n, & \text{otherwise}. \end{cases} \tag{3.4}$$

Since θ is a *-homomorphism, $\|\theta_n(xy) - \theta_n(x)\theta_n(y)\| \to 0$ for all $x, y \in A$; from this it follows that $\alpha_{m,n}$ is also a *-homomorphism.

The first equation of (3.3) implies that $\lim_n \|\overline{\gamma}_n(x) - \theta_n(x)\| = 0$ for all $x \in A$, which in turn implies that

$$\|\alpha_m(x) - \pi_{I\cap J}(\overline{\gamma}(x))\| = \sup_{n\geq m} \|\overline{\gamma}_n(x) - \theta_n(x)\| \to 0 \tag{3.5}$$

for all $x \in A$. Thus, $(\alpha_m)_m$ converges in the point-norm topology to the liftable *-homomorphism $\pi_{I\cap J} \circ \overline{\gamma}$, and since A is ℓ-open, it follows that α_m is liftable for some sufficiently large m. Let $\beta = (\beta_n)_{n\in\Lambda} : A \to B$ be a lift of α_m, where $\beta_n : A \to B_n$ is a *-homomorphism for each n. The fact that β is a lift amounts to

$$(\beta_n(x) - \alpha_{m,n}(x))_{n\in\Lambda} \in I \cap J, \quad \text{for all } x \in A. \tag{3.6}$$

This implies first that $\lim_n \|\beta_n(x) - \theta_n(x)\| = 0$ for all $x \in A$, and combining this with the first equation of (3.3), it follows that

$$\lim_n \|\beta_n(x) - \overline{\gamma}_n(x)\| = 0, \quad \text{for all } x \in A. \tag{3.7}$$

From (3.6), we also get that $\pi_{I_n} \circ \beta_n(x) - \pi_{I_n} \circ \theta_n$ for all $n \geq m$, and combining this with the second equation of (3.3), we have that β_n is a lift of φ_n for $n \geq m$. In summary, for sufficiently large n we find that β_n is a lift of φ_n which is point-norm close to $\overline{\gamma}_n$, in contradiction to our initial assumption. □

We now pick up some consequences, using ideas from of Blackadar [3]. We add the proofs for completion. The first tells us that when A is ℓ-open, $\mathrm{Hom}(A, B)$ is locally path-connected in a uniform way.

Corollary 3.2 (cf. [3, Corollary 4.2]**).** *Let A be an ℓ-open C^*-algebra (or more generally, one that satisfies the conclusion of Theorem* 3.1*). For any $\epsilon > 0$ and any finite set $\mathcal{F} \subset A$, there is a $\delta > 0$ and a finite set $\mathcal{G} \subset A$ such that whenever B is a C^*-algebra, φ_0 and φ_1 are *-homomorphisms from A to B/I with $\|\varphi_0(u) - \varphi_1(u)\| < \delta$ for all $u \in \mathcal{G}$, then there is a point-norm continuous path $(\varphi_t)_{t\in[0,1]}$ of *-homomorphisms from A to B connecting φ_0 and φ_1 with $\|\varphi_0(v) - \varphi_t(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $t \in [0, 1]$. In particular, $Hom(A, B)$ is locally path-connected for any C^*-algebra B.*

Proof. For any $\epsilon > 0$ and finite set $\mathcal{F}$, choose $\delta > 0$ and finite set $\mathcal{G}$ as in Theorem 3.1. Let $D := C([0, 1], B)$ and $I := C_0((0, 1), B)$. Then $D/I \cong B \oplus B$. Define *-homomorphisms $\gamma, \varphi : A \to D/I$ by $\gamma(x) := (\varphi_0(x), \varphi_0(x))$ and $\varphi(x) := (\varphi_0(x), \varphi_1(x))$. Then γ lifts to a *-homomorphism $\mathrm{id}_{C([0,1])} \otimes \varphi_0 : A \to D$, and so these two maps satisfy the hypothesis of Theorem 3.1. Hence the conclusion of Theorem 3.1 holds and there exists a *-homomorphism $\overline{\varphi} = (\overline{\varphi}_t)_{t\in[0,1]} : A \to D$ such that

$$\|\overline{\gamma}(a) - \overline{\varphi}(a)\| < \epsilon \quad \text{for all } a \in \mathcal{F}. \tag{3.8}$$

Then $\overline{\varphi}$ is a homotopy of *-homomorphisms $A \to B$ connecting φ_0 to φ_1, and (3.8) tells us that $\|\varphi_t(a) - \varphi_0(a)\| < \epsilon$ for all $a \in \mathcal{F}$, as required. □

Example 3.3. Consider the topologist' sine curve:

$$X := \{(x, y) : y = \sin(\frac{\pi}{x}), 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}. \tag{3.9}$$

Then $\mathrm{Hom}(C(X), \mathbb{C}) = X$, which is not locally path-connected; therefore by the above corollary, $C(X)$ is not ℓ-open.

Theorem 3.4 (Homotopy Lifting Theorem; cf. [3, Theorem 5.1]**).** *Let A be an ℓ-open C^*-algebra (or more generally, one that satisfies the conclusion of Theorem* 3.1*). Let B be a C^*-algebra, I a closed ideal of B, $(\varphi_t)_{t\in[0,1]}$ a point-norm continuous path of *-homomorphisms from A to B/I. Suppose φ_0 lifts to a *-homomorphism $\overline{\varphi_0} : A \to B$. Then there is a point-norm continuous path $(\overline{\varphi_t})_{t\in[0,1]}$ of *-homomorphisms from A to B starting at $\overline{\varphi_0}$ such that $\overline{\varphi_t}$ is a lift of φ_t for all $t \in [0, 1]$.*

Proof. Take an arbitrary finite set $\mathcal{F}$ of A and real number $\epsilon > 0$, and let $\mathcal{G}, \delta$ be given by Theorem 3.1. We can find a partition $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$ such that $\|\varphi_t(a) - \varphi_s(a)\| < \delta$ for all $a \in \mathcal{G}$ whenever $t, s \in [t_{i-1}, t_i]$, for any i.

Let $D := C([0, t_1], B)$ and $J := C_0((0, t_1], I)$, which is an ideal of D, so that

$$\begin{aligned} D/J &\cong C([0, t_1] : B/I) \oplus_{\pi_I} B \\ &= \{(f, b) \in C([0, t_1] : B/I) \oplus B : f(0) = \pi_I(b)\}. \end{aligned} \tag{3.10}$$

Making this identification, define *-homomorphisms $\gamma := (\mathrm{id}_{C([0,t_1])} \otimes \varphi_0) \oplus \overline{\varphi}_0, \theta := \varphi|_{[0,t_1]} \oplus \overline{\varphi}_0 : A \to D/J$ (where $\varphi|_{[0,t_1]}$ denotes the *-homomorphism $A \to C([0, t_1], B/I)$ given by restricting the homotopy (φ_t) to $[0, t_1]$). Then γ lifts to the *-homomorphism $\mathrm{id}_{C([0,t_1])} \otimes \overline{\varphi}$, so by Theorem 3.1, φ lifts, giving a continuous path of lifts $(\overline{\varphi_t})$ of (φ_t) for $t \in [0, t_1]$. Continuing the same process for successive intervals $[t_1, t_2], \ldots, [t_{n-1}, t_n]$, we get the required continuous path $(\overline{\varphi_t})_{t\in[0,1]}$, such that $\overline{\varphi}_t$ lifts φ_t for all $t \in [0, 1]$. □

Proposition 3.5. *Let A be an unital C^*-algebra. Then A satisfies the conclusion of the Homotopy Lifting Theorem if and only if A satisfies the conclusion in the category of unital C^*-algebras and unital *-morphisms.*

Proof. Suppose A satisfies the conclusion of the Homotopy Lifting Theorem in the category of unital C^*-algebras and unital *-morphisms. Let B be a C^*-algebra, I a closed ideal of B, $(\varphi_t)_{t\in[0,1]}$ a point-norm continuous path of *-homomorphisms from A to B/I, and $\overline{\varphi_0} : A \to B$ a lift of φ_0. Set $q_0 := \varphi_0(1), q_1 := \varphi_1(1)$, and $p_0 := \overline{\varphi_0}(1)$. Then, q_0 is homotopic to q_1. Since $\mathbb{C}$ is ℓ-open, Theorem 3.4 implies that there exists a continuous path of projections $(p_t)_{t\in[0,1]}$ connecting p_0 and p_1 with $q_1 = \pi_I(p_1)$. Consequently, we can find a continuous path of partial isometries $(v_t)_{t\in[0,1]}$ such that

$$\begin{aligned} v_0 &= p_0, \\ v_t^* v_t &= p_0 \ \forall\ t, \\ v_t v_t^* &= p_t. \end{aligned} \tag{3.11}$$

Let $\psi_1 := \pi_I(v_1^*)\varphi_1\pi_I(v_1) : A \to q_0(B/I)q_0$. Then, $(\pi_I(v_t^*)\varphi_t\pi_I(v_t))_{t\in[0,1]}$ is a point-norm continuous paths of unital *-homomorphisms from A to $q_0(B/I)q_0$. Using the conclusion of the Homotopy Lifting Theorem in the unital category, ψ_1 lifts to a unital *-homomorphism $\overline{\alpha_1} : A \to p_0Bp_0$ and there is a point-norm continuous path $(\overline{\alpha_t})_{t\in[0,1]}$ of unital *-homomorphisms connecting φ_0 to $\overline{\alpha_1}$. Moreover, $\overline{\alpha_t}$ is a lift of $\pi_I(v_t^*)\varphi_t\pi_I(v_t)$ for each $t \in [0, 1]$. Set $\overline{\varphi_t} := v_t\alpha_t v_t^* : A \to B$. Then, $(\overline{\varphi_t})_{t\in[0,1]}$ defines a point-norm continuous path of *-homomorphisms from A to B starting at $\overline{\varphi_0}$ such that $\overline{\varphi_t}$ is a lift of φ_t for all $t \in [0, 1]$. The proof of the converse follows directly from the statement. □

Example 3.6. Using Proposition 3.5 and [18, Theorem 3.5], AF-algebras satisfy the condition of the Homotopy Lifting Theorem.

Remark 3.7. Conway ([7, 8]) studied a restricted version of the homotopy lifting theorem, which he called the C^*-covering homotopy property. He considered Theorem 3.4 in the case where B/I is the Calkin algebra.

Combining all the previous theorems and corollaries, we have the following characterization of ℓ-open C^*-algebra.

Theorem 3.8. *Let A be a C^*-algebra. Then the following are equivalent*

(i) *A is ℓ-open.*
(ii) *The system of maps* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/J)$ *(over all C^*-algebras B and ideals J) is uniformly open, as in the conclusion of Theorem* 3.1
(iii) *A satisfies the conclusion of the Homotopy Lifting Theorem (Theorem* 3.4*) and* $\mathrm{Hom}(A, B)$ *is locally path-connected for all C^*-algebras B.*

Proof. (i)⇒(ii) is Theorem 3.1 and (ii)⇒(iii) is by Corollary 3.2 and Theorem 3.4.

To prove that (iii)⇒(i), let $\phi_n : A \to B/I$ be a net of *-homomorphisms which converges point-norm to a liftable *-homomorphism $\phi : A \to B/I$. Since $\mathrm{Hom}(A, B/I)$ is locally path-connected, ϕ_n is homotopic to ϕ for sufficiently large n. The conclusion of the Homotopy Lifting Theorem then implies that ϕ_n is liftable for these n. This shows that $\mathrm{Hom}(A, B, I)$ is open in $\mathrm{Hom}(A, B/I)$, as required. □

Example 3.9. Satisfying the condition of the Homotopy Lifting Theorem doesn't guarantee ℓ-openness of C^*-algebras. M_{2^∞} satisfies the condition of the Homotopy Lifting Therem (see Example 3.6), but it is not an ℓ-open C^*-algebra. To see that M_{2^∞} is not ℓ-open, suppose otherwise. Using any finite set $\mathcal{F} \subseteq M_{2^\infty}$ and any $\epsilon > 0$, obtain $\delta > 0$ and a finite set $\mathcal{G} \subset M_{2^\infty}$ according to Theorem 3.1. Without loss of generality, we can assume $\mathcal{G} \subset M_{2^k}$ for some k.

Let us set $B := B(\mathcal{H})$ and $J := \mathcal{K}$, so that B/J is the Calkin algebra. Let $\phi_1, \phi_2 : A \to B/J$ be *-homomorphisms such that ϕ_1 is liftable but ϕ_2 is not (which exists by [20]). Define $\varphi_i := \mathrm{id}_{M_{2^k}} \otimes \phi_i : M_{2^k} \otimes M_{2^\infty} \cong M_{2^\infty} \to M_{2^k} \otimes (B/J) \cong (M_{2^k} \otimes B)/(M_{2^k} \otimes J)$. Then we have that $\varphi_1(a) = \varphi_2(a)$ for all $a \in \mathcal{G}$. Hence, Theorem 3.1 tells us that since φ_1 is liftable, so is φ_2. The Ext-class of φ_2 is 2^k times the Ext-class of ϕ_2; $Ext(M_{2^\infty})$ is the 2-adic integers, which is torsion-free, it follows that φ_2 is not liftable, a contradiction. Hence, M_{2^∞} is not ℓ-open.

The characterization of ℓ-openness confirms a conjecture of Blackadar [3, Page 299], as follows.

Corollary 3.10. *Let A be an ℓ-open C^*-algebra. Then A is ℓ-closed.*

Proof. Fix a $\epsilon > 0$ and a finite set $\mathcal{F}$ and choose a $\delta > 0$ and finite set $\mathcal{G}$ as in Theorem 3.1. Let $\phi_n : A \to B/I$ be a net of liftable *-homomorphisms which converges point-norm to a *-homomorphism $\phi : A \to B/I$. We can find m such that $\|\phi_m(u) - \phi(u)\| < \delta$ for all $u \in \mathcal{G}$. Since ϕ_m is liftable, the conclusion of Theorem 3.1 implies that ϕ is liftable. Hence, A is ℓ-closed. □

4. Characterization of ℓ-closed C^*-algebras

We now characterize ℓ-closed C^*-algebras, showing that the condition is equivalent to a uniform relative openness of the map $\mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B/I)$. We require separability for this characterization, and one direction uses a Cauchy sequence argument.

Theorem 4.1. *Let A be a separable C^*-algebra. Then the following are equivalent:*

(i) *A is ℓ-closed.*
(ii) *For any $\epsilon > 0$ and finite set $\mathcal{F} \subset A$, there is a $\delta > 0$ and a finite set $\mathcal{G} \subset A$ such that whenever B is a C^*-algebra, I is a closed ideal of B, ψ and ϕ are *-homomorphisms from A to B with $\|\pi_I \circ \phi(u) - \pi_I \circ \psi(u)\| < \delta$ for all $u \in \mathcal{G}$, then there exists a *-homomorphism $\eta : A \to B$ such that $\|\phi(v) - \eta(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \psi = \pi_I \circ \eta$.*

Proof. (i)$\Rightarrow$(ii). Let $(\mathcal{G}_n)$ be an increasing sequence of finite subsets of A whose union is dense in A. Suppose (ii) is false for a fixed $\epsilon > 0$ and finite set $\mathcal{F} \subset A$. Then, there are C^*-algebras B_n with ideals I_n, and *-homomorphisms $\phi_n, \psi_n : A \to B_n$ such that

$$\|\pi_{I_n} \circ \phi_n(a) - \pi_{I_n} \circ \psi_n(a)\| < \frac{1}{n} \quad \text{for all } a \in \mathcal{G}_n, \tag{4.1}$$

but no *-homomorphism $\eta_n : A \to B_n$ satisfies both $\|\phi_n(a) - \eta_n(a)\| < \epsilon$ for all $a \in \mathcal{F}$ and $\pi_{I_n} \circ \psi_n = \pi_{I_n} \circ \eta_n$.

Let $B := \prod_{n=1}^{\infty} B_n$, $I := \prod_{n=1}^{\infty} I_n$, and $J := \bigoplus_{n=1}^{\infty} B_n$. Define *-homomorphisms $\overline{\phi} := (\phi_1, \phi_2, \dots), \overline{\psi} := (\psi_1, \psi_2, \dots) : A \to B$.

By (4.1), it follows that $\pi_{I+J} \circ \overline{\phi} = \pi_{I+J} \circ \overline{\psi}$. Then by the general Chinese remainder theorem (Lemma 2.4), there exists a *-homomorphism $\theta : A \to B/(I \cap J)$ such that

$$\pi_J \circ \overline{\phi} = \pi_J \circ \theta \ \text{ and } \pi_I \circ \overline{\psi} = \pi_I \circ \theta \tag{4.2}$$

For each $n \in \mathbb{N}$, define the *-homomorphism

$$\overline{\alpha}_n := (\psi_1, \psi_2, \dots, \psi_{n-1}, \phi_n, \phi_{n+1}, \dots) : A \to B. \tag{4.3}$$

Then by the definition of J, we have $\pi_J \circ \overline{\alpha}_n = \pi_J \circ \overline{\phi}$. Therefore by (4.2), for $x \in A$,

$$\begin{aligned} \|\pi_{I\cap J} \circ \overline{\alpha}_n(x) - \theta(x)\| &= \|\pi_I \circ \overline{\alpha}_n(x) - \pi_I \circ \overline{\psi}(x)\| \\ &= \sup_{m \geq n} \|\pi_{I_m} \circ \phi_m(x) - \pi_{I_m} \circ \psi_m(x)\| \to 0. \end{aligned} \tag{4.4}$$

Since A is ℓ-closed, we deduce that θ lifts to a *-homomorphism $\eta = (\eta_1, \eta_2, \dots) : A \to B$. Then (4.2) implies that $\pi_{I_n} \circ \psi_n = \pi_{I_n} \circ \eta_n$ and $\lim_{n\to\infty} \|\phi_n(x) - \eta_n(x)\| = 0$ for all $x \in A$. Hence, there is a k such that

$$\|\phi_k(a) - \eta_k(a)\| < \epsilon \tag{4.5}$$

for all $a \in \mathcal{F}$. This is a contradiction.

(ii)⇒(i). Suppose $\eta_n : A \to B/I$ is a sequence of liftable *-homomorphisms which converges pointwise to a *-homomorphism $\eta : A \to B/I$. Let $\mathcal{F}_n$ be an increasing sequence of finite sets whose union is dense in A. Choose $\delta_n > 0$ and a finite set $\mathcal{G}_n$ such that they satisfy the conditions of (ii) with $\epsilon := \frac{1}{2^n}$ and $\mathcal{F} := \mathcal{F}_n$. By passing to a subsequence, we may assume without loss of generality that

$$\|\eta_n(u) - \eta_{n+1}(u)\| < \delta_n \quad \text{for all } u \in \mathcal{G}_n. \tag{4.6}$$

Let $\overline{\eta}_n : A \to B$ be a lift of η_n. Then by the choice of $\mathcal{G}_1$ and δ_1 from (ii) implies that there exists a *-homomorphism $\xi_2 : A \to B$ such that $\|\overline{\eta}_1(v) - \xi_2(v)\| < \frac{1}{2}$ for all $v \in \mathcal{G}_1$ and $\pi_I \circ \overline{\eta}_2 = \pi_I \circ \xi_2$. Then we have $\|\pi_I \circ \overline{\eta}_2(u) - \pi_I \circ \overline{\eta}_3(u)\| = \|\pi_1 \circ \xi_2(u) - \pi_I \circ \overline{\eta}_3(u)\| < \delta_2$ for all $u \in \mathcal{G}_2$. Using the choice of $\mathcal{G}_2$ and δ_2 from (ii), we have a *-homomorphism $\xi_3 : A \to B$ such that $\|\xi_2(v) - \xi_3(v)\| < \frac{1}{2^2}$ and $\pi_I \circ \overline{\eta}_3 = \pi_I \circ \xi_3$. Continuing the process and setting $\xi_1 = \overline{\eta}_1$, we get a $(\xi_n : A \to B)$ such that $\|\xi_n(a) - \xi_{n+1}(a)\| < \frac{1}{2^n}$ for all $a \in F_n$ and $\eta_n = \pi_I \circ \xi_n$. Consequently, the sequence $(\xi_n(a))_{n=1}^{\infty}$ is Cauchy for each $a \in A$, so it converges to some $\xi(a) \in B$. This defines a *-homomorphism $\xi : A \to B$, and for $a \in A$,

$$\pi_I \circ \xi(a) = \lim_n \pi_I \circ \xi_n(a) = \lim_n \eta_n(a) = \eta(a). \tag{4.7}$$

Therefore we obtain a lift of η, and this shows that A is ℓ-closed. □

Note that condition (iii) of Theorem 3.8 strengthens condition (ii) in Theorem 4.1, by replacing $\psi : A \to B$ with a map $A \to B/I$ which is (a priori) not liftable. This gives a quick proof of Corollary 3.10 in the separable case.

Theorem 4.1 may be reformulated as follows.

Theorem 4.2. *Let A be a separable C^*-algebra and S a generating set of A. Then the following are equivalent:*

(i) *A is ℓ-closed.*
(ii) *For any $\epsilon > 0$ and finite set $\mathcal{F} \subset S$, there is a $\delta > 0$ and a finite set $\mathcal{G} \subset S$ such that whenever B is a C^*-algebra, I is a closed ideal of B, ψ and ϕ are *-homomorphisms from A to B with $\|\pi_I \circ \phi(u) - \pi_I \circ \psi(u)\| < \delta$ for all $u \in \mathcal{G}$, then there exists a *-homomorphism $\eta : A \to B$ such that $\|\phi(v) - \eta(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \psi = \pi_I \circ \eta$.*

In [3, Example 6.4], Blackadar asks whether $C^*(F_\infty)$, the universal C^*-algebra generated by a sequence of unitaries, is ℓ-closed. We now show that it is.

Example 4.3. $C^*(F_\infty)$ is ℓ-closed. To see this, consider $\epsilon > 0$, a finite set $\mathcal{F} \subset \{u_1, u_2, \ldots\}$, an ideal I of B, and *-homomorphisms $\phi, \psi : C^*(F_\infty) \to B$. Without loss of generality, we may assume $F = \{u_1, u_2, \ldots, u_n\}$ for some n. $C^*(F) \cong C^*(F_n)$ is semiprojective ((this is well-known; see [1, Corollary 2.22 and Proposition 2.31] for example) and so ℓ-closed by [3, Corollary 6.2]. Choose $\delta > 0$ and a finite set $\mathcal{G} \subset \mathcal{F}$ as in Theorem 4.2 (applied to $C^*(F)$).

Then $\|\pi_I \circ \phi(u) - \pi_I \circ \psi(u)\| < \delta$ for all $u \in \mathcal{G}$ implies there exists a *-homomorphism $\xi : C^*(F_n) \to B$ such that $\|\phi(v) - \xi(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \xi = \pi_I \circ \psi|_{C^*(F_n)}$. $\eta : C^*(F_\infty) \to B$ defined by

$$\eta(u_m) := \begin{cases} \xi(u_m), & m \leq n; \\ \psi(u_m), & m > n \end{cases} \tag{4.8}$$

is a *-homomorphism satisfying $\|\phi(v) - \eta(v)\| < \epsilon$ for all $v \in \mathcal{F}$ and $\pi_I \circ \xi = \pi_I \circ \eta$, as required.

5. Commutative unital ℓ-open C^*-algebras

In this section, we show that commutative unital separable ℓ-open C^*-algebras coincide with commutative unital separable semiprojective C^*-algebras. We begin with the following which may be of independent interest.

Proposition 5.1. *Let A be an ℓ-open C^*-algebra and $\psi : A \to B$ a weakly semiprojective *-homomorphism. Then ψ is a semiprojective *-homomorphism.*

Proof. Fix $\epsilon > 0$ and a finite set $\mathcal{F}$ in A, and let $\delta > 0$ and $\mathcal{G} \subset A$ be given by Theorem 3.1. Given any *-homomorphism $\varphi : B \to C/\overline{\bigcup_n J_n}$ with $J_1 \lhd J_2 \lhd \cdots \lhd C$ an increasing sequence of closed ideals of a C^*-algebra C, by weak semiprojectivity we can find some n and a *-homomorphism $\phi : A \to C/J_n$ such that

$$\|\varphi \circ \psi(u) - \pi \circ \phi(u)\| < \delta \tag{5.1}$$

for all $u \in \mathcal{G}$. It follows from Theorem 3.1 that there exists a *-homomorphism $\rho : A \to C/J_n$ such that $\varphi \circ \psi = \pi \circ \rho$. Hence ψ is a semiprojective *-homomorphism. ☐

Lemma 5.2 ([6], Proposition 3.1). *Let X be a compact, connected, and locally connected metric space, of covering dimension > 1. Then X contains a topological copy of the circle S^1.*

Theorem 5.3. *Let X be a compact metric space. Then the following are equivalent*

(i) *$C(X)$ is a semiprojective C^*-algebra.*
(ii) *$C(X)$ is an ℓ-open C^*-algebra.*
(iii) *X is an ANR and* $\dim(X) \leq 1$.

Proof. (i)⇒(ii) follows from [3, Corollary 6.2] and (iii)⇒(i) follows from [19, Theorem 1.2]. We prove that (ii)⇒(iii), along the lines of Sørensen and Thiel's proof of [19, Proposition 3.1].

Suppose $C(X)$ is ℓ-open. Blackadar showed that X is e-open and thus locally contractible [2, Corollary 4.3]. The Homotopy Lifting Theorem (Theorem 3.4) implies the homotopy extension theorem for X; since X is also locally contractible, we have that X is an ANR by [13, Theorem IV.2.4].

Suppose by contradiction that $\dim(X) \geq 2$. Since X is compact, we have that $\operatorname{locdim}(X) = \dim(X) \geq 2$, which implies that there is an $x_0 \in X$ such that $\dim(D) \geq 2$ for every closed neighbourhood D of x_0 (see [16] for details on $\operatorname{locdim}(X)$). Let $D_1, D_2, \ldots$ be a decreasing sequence of closed neighbourhoods of x_0 with $\dim(D_k) \geq 2$ for all k. Using Lemma 5.2, there exists a topological embedding $\psi_k : S^1 \hookrightarrow D_k \subset X$ for each k. Let

$$Y := (0,0) \cup \bigcup_{k \geq 1} S\Big(\big(\frac{1}{2^k}, 0\big), \frac{1}{4 \cdot 2^k}\Big) \subset \mathbb{R}^2. \tag{5.2}$$

Then $C(Y)$ is weakly semiprojective ([19]]). Define $\psi : Y \to X$ to send $(0,0)$ to x_0 and to be ψ_k on the circle $S((\frac{1}{2^k}, 0), \frac{1}{4.2^k})$. Then ψ induces a *-homomorphism $\psi^* : C(X) \to C(Y)$, which is weakly semiprojective since $C(Y)$ is.

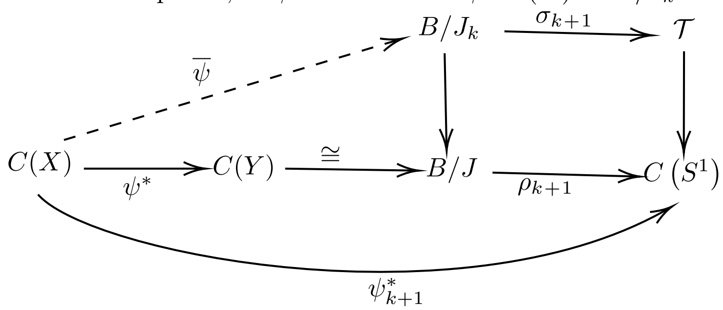
Let $\mathcal{T}$ be the Toeplitz algebra and let $\mathcal{K}$ be the ideal of compact operators. Writing $A^\dagger$ for the unitization of A, set

$$\begin{aligned} B &:= \Big(\bigoplus_{k \geq 1} \mathcal{T}\Big)^\dagger \\ &= \{(t_1, t_2, \ldots,) \in \prod_{k \geq 1} \mathcal{T} : (t_k)_k \text{ converges to a scalar multiple of } 1_{\mathcal{T}}\} \end{aligned} \tag{5.3}$$

and $J_k := \underbrace{\mathcal{K} \oplus \mathcal{K} \oplus \cdots \mathcal{K}}_{k\ times} \oplus 0 \oplus 0 \cdots$. Then $J_k \subset J_{k+1}$, $J = \overline{\bigcup_k J_k} = \bigoplus_{k \geq 1} \mathcal{K}$,

$$B/J_k = \underbrace{C(S^1) \oplus C(S^1) \oplus \cdots \oplus C(S^1)}_{k\ times} \oplus \Big(\bigoplus_{l \geq k+1} \mathcal{T}\Big)^\dagger, \tag{5.4}$$

and $B/J = (\bigoplus_{k \geq 1} (C(S^1)))^\dagger \cong C(Y)$. Proposition 5.1 implies ψ^* is a semiprojective *-homomorphism, so ψ^* lifts to some $\overline{\psi} : C(X) \to B/J_k$.



Let $\sigma_{k+1} : B/J_k \to \mathcal{T}$ be the projection of B/J_k onto the (k+1)-th coordinate and $\rho_{k+1} : B/J \to C(S^1)$ be the projection of B/J onto the (k+1)-th coordinate. Note that $\rho_{k+1} \circ \psi^* : C(X) \to C(S^1)$ coincide with the *-homomorphism induced by $\psi_{k+1} : S^1 \hookrightarrow D_{k+1} \subset X$ and it is surjective since ψ_{k+1} is an inclusion. The generating unitary of $C(S^1)$ lifts to a normal element in $C(X)$ under ψ^*_{k+1}, but it does not lift to a normal element in $\mathcal{T}$, which is a contradiction. Hence, $\dim(X) \leq 1$. $\square$

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