# Nuclear dimension and $\mathcal{Z}$-stability of simple C*-algebras

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Joint work with Jorge Castillejos, Sam Evington, Stuart White, and Wilhelm Winter.

# Outline

- Background/motivation – classification of C*-algebras, funny examples.
- The Toms–Winter conjecture.
- The Jiang–Su algebra $\mathcal{Z}$, and $\mathcal{Z}$-stability.
- Nuclear dimension.
- Main result and applications.

# Background: classification of C*-algebras

**Conjecture (Elliott, early '90s)**

Simple separable nuclear C*-algebras are classified by K-theory and traces.

If $A, B$ are two simple separable nuclear C*-algebras, and there is an isomorphism of their invariants (consisting of ordered K-theory, traces, and the pairing between $K_0$ and traces) then

$$A \cong B.$$

More generally, it was hoped that the structure of nuclear C*-algebras would mimic the nice structure of von Neumann algebras, with just a bit of extra topological bookkeeping.

# Background: classification of C*-algebras

Examples of unusual C*-algebras soon appeared.

- Villadsen, '98: A simple separable unital nuclear (in fact, AH) C*-algebra such that the order on $K_0$ is ~~un~~perforated.

  ∃ proj's $p, q$, $\tau(p) < \tau(q)$ but $p \not\lesssim q$

- Rørdam, '97: A simple separable unital nuclear (in fact, AH) C*-algebra $A$ such that $A$ is not stable but $M_2(A)$ is.
- Villadsen, '99: A simple separable unital nuclear (in fact, AH) C*-algebra which is finite but has stable rank $> 1$.
- Rørdam, '03: A simple separable unital nuclear C*-algebra which contains both an infinite and a finite projection.

  NOT like a $\mathrm{II}_\infty$ vN alg. ∃ proj $p$ that is finite but $p \oplus p$ is infinite.

- Toms, '08: A pair of simple separable unital nuclear (in fact, AH) C*-algebras $A, B$ which refute the Elliott conjecture.

# The Toms–Winter regularity conjecture

Goal: separate the wheat from the chaff, within the silo of simple nuclear C*-algebras.

**Conjecture (Toms and Winter, late ’00s)**

Among simple, separable, nuclear, unital C*-algebras, the following properties coincide:

(i) Strict comparison of positive elements. ← used by Toms to show $A \not\cong B$

(ii) $\mathcal{Z}$-stability ($A \cong A \otimes \mathcal{Z}$ where $\mathcal{Z}$ is the Jiang–Su algebra).

(iii) Finite nuclear dimension ($\dim_{nuc} A < \infty$).

# The Jiang–Su algebra, $\mathcal{Z}$

Consider first the CAR algebra:

$$M_2 = \left(\begin{array}{c|c} \mathbb{C} & \mathbb{C} \\ \hline \mathbb{C} & \mathbb{C} \end{array}\right)$$

# The Jiang–Su algebra, $\mathcal{Z}$

Consider first the CAR algebra:

$$M_4 = \left( \begin{array}{c|c} \begin{array}{c|c} \mathbb{C} & \mathbb{C} \\ \hline \mathbb{C} & \mathbb{C} \end{array} & \begin{array}{c|c} \mathbb{C} & \mathbb{C} \\ \hline \mathbb{C} & \mathbb{C} \end{array} \\ \hline \begin{array}{c|c} \mathbb{C} & \mathbb{C} \\ \hline \mathbb{C} & \mathbb{C} \end{array} & \begin{array}{c|c} \mathbb{C} & \mathbb{C} \\ \hline \mathbb{C} & \mathbb{C} \end{array} \end{array} \right)$$

Consider first the CAR algebra:

$$M_8 = \left( \begin{array}{c|c|c|c|c|c|c|c}
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \hline
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \hline
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \hline
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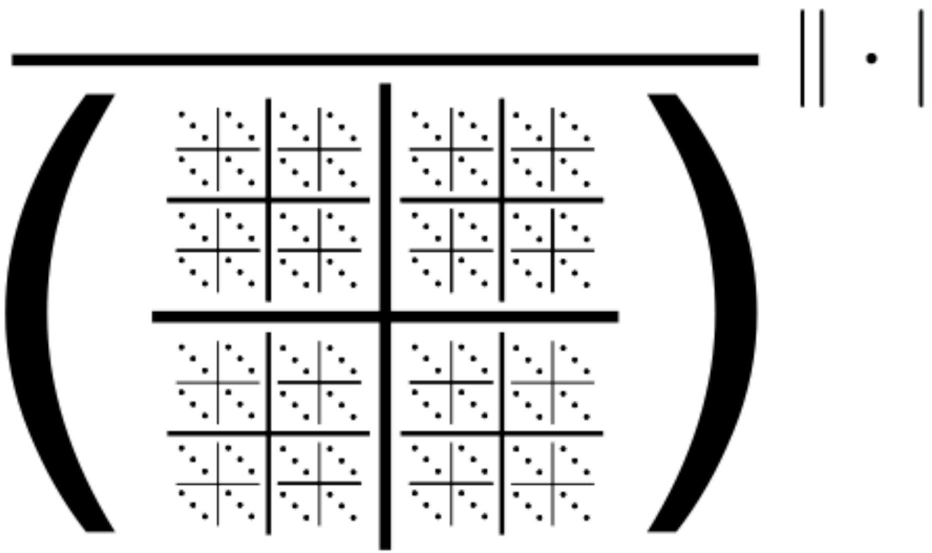
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C}
\end{array} \right)$$

Consider first the CAR algebra:

$M_{2^\infty} =$

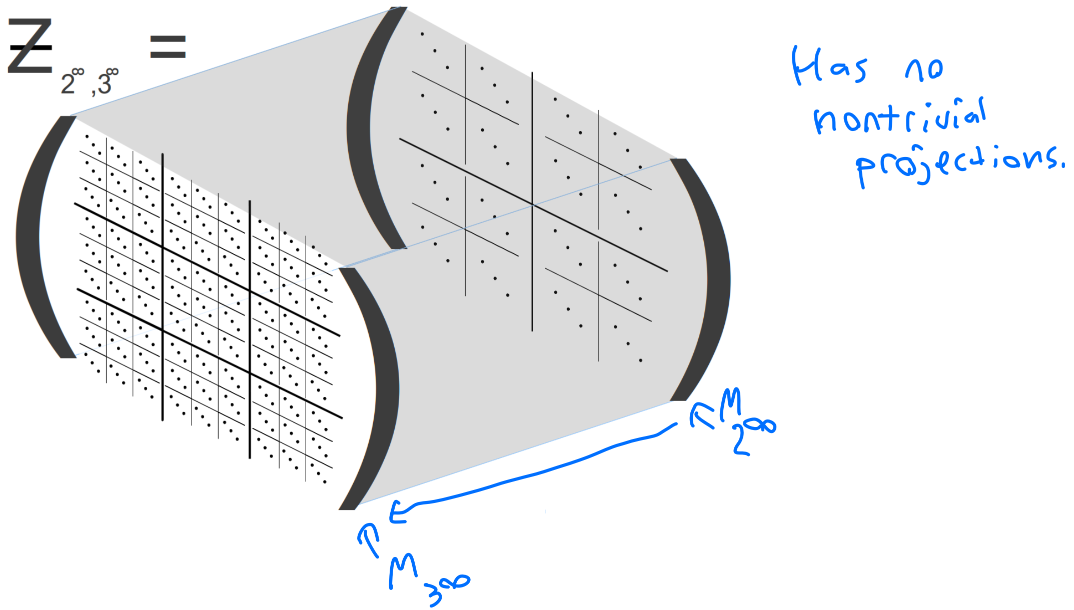


$$M_{2^\infty} = \overline{M_2(\mathbb{C})^{\otimes\infty}}^{\|\cdot\|}$$

$M_{3^\infty} \not\cong M_{2^\infty}$

$M_{3^\infty} \otimes M_{2^\infty} \not\cong M_{3^\infty}$

# The Jiang–Su algebra, $\mathcal{Z}$

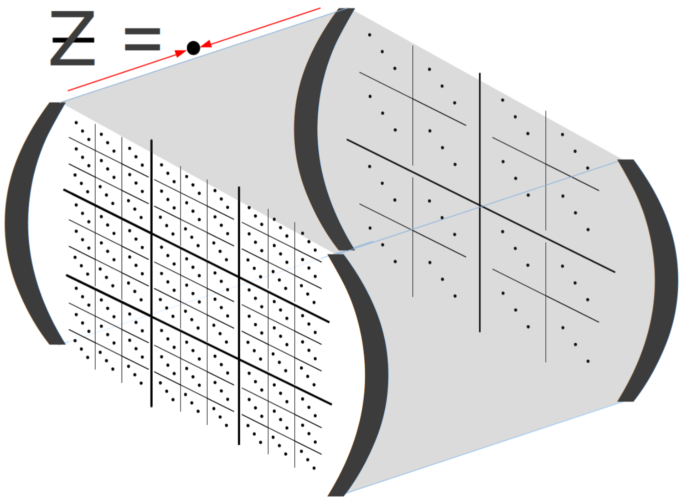



$$\mathcal{Z}_{2^\infty,3^\infty} := \{f \in C([0,1] \mid M_{2^\infty} \otimes M_{3^\infty}) \mid \; f(0) \in 1_{2^\infty} \otimes M_{3^\infty}, \\ f(1) \in M_{2^\infty} \otimes 1_{3^\infty}\}$$

$$\mathcal{Z} := \varinjlim \mathcal{Z}_{2^\infty,3^\infty}.$$

Simple, unique trace, $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}^{\otimes\infty}$.

# The Jiang–Su algebra, $\mathcal{Z}$




$$\mathcal{Z}_{2^\infty,3^\infty} := \{f \in C([0,1] \mid M_{2^\infty} \otimes M_{3^\infty}) \mid \ f(0) \in 1_{2^\infty} \otimes M_{3^\infty}, \\ f(1) \in M_{2^\infty} \otimes 1_{3^\infty}\}$$

$$\mathcal{Z} := \varinjlim \mathcal{Z}_{2^\infty,3^\infty}.$$

Very nontrivial!

Simple, unique trace, $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}^{\otimes\infty}$.

# Jiang–Su stability

$A$ is *$\mathcal{Z}$-stable* if $A \cong A \otimes \mathcal{Z}$.

**Theorem (McDuff, Kirchberg, Rørdam, Toms–Winter, Dadarlat–Toms)**

Let $A$ be a unital C*-algebra. The following are equivalent.

(i) $A$ is $\mathcal{Z}$-stable.

(ii) $\mathcal{Z}$ embeds unitally into $A_\infty \cap A'$ (where $A_\infty := l_\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$).

(iii) There is a subhomogeneous C*-algebra with no characters, which embeds unitally into $A_\infty \cap A'$.

# Nuclear dimension

Partition of unity
(topology)

↙ ↘

Noncommutative POU = c.p. approx. prop.

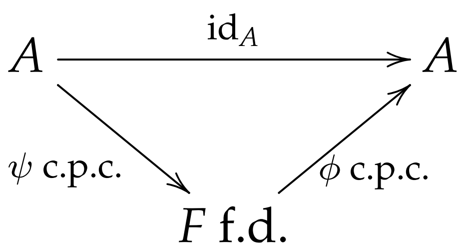
Lebesgue covering dimension

↘ ↙

Nuclear dimension

# Nuclear dimension

Completely positive approximation property:

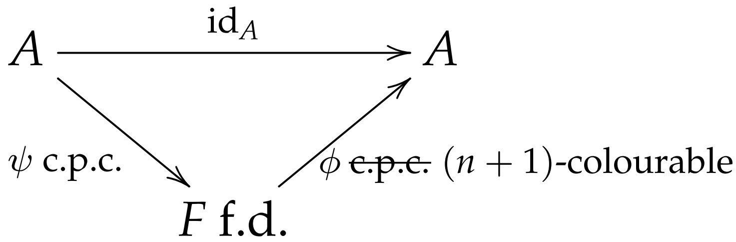



(Equivalent to nuclearity of A.)

approximately commuting in point-$\|.\|$, i.e., $\|\phi(\psi(a)) - a\| < \epsilon$ for all $a$ in a finite set $F$.

# Nuclear dimension

Nuclear dimension at most $n$ ($\dim_{nuc} A \leq n$) (Kirchberg–Winter '04, Winter–Zacharias '10):




approximately commuting in point-$\|.\|$, i.e., $\|\phi(\psi(a)) - a\| < \epsilon$ for all $a$ in a finite set $F$.

$\phi$ is $(n+1)$-colourable: $F = F_0 \oplus \cdots \oplus F_n$ such that $\phi|_{F_i}$ is c.p.c. and orthogonality-preserving (aka order zero).

$xy = 0 \quad x, y \in F_i$

$\phi(x)\phi(y) = 0$

Eg. $\dim_{nuc} C(X) = \dim X$.

# Equivalence of finite nuclear dimension and $\mathcal{Z}$-stability

**Theorem**

If $A$ is a simple separable nuclear C*-algebra, then $A$ is $\mathcal{Z}$-stable iff $\dim_{nuc} A < \infty$. ($\Leftrightarrow$ $\dim_{nuc} A \leq 1$)

- Winter, '10, '12: $\dim_{nuc} A < \infty \Rightarrow A \cong A \otimes \mathcal{Z}$.
- Using Kirchberg–Phillips classification: $A \cong A \otimes \mathcal{Z} \Rightarrow \dim_{nuc} A < \infty$, if $T(A) = \emptyset$. (A is infinite)
- Matui–Sato, '14: $A \cong A \otimes \mathcal{Z} \Rightarrow \dim_{nuc} A < \infty$, assuming unique trace and quasidiagonality.
- Sato–White–Winter '15: $A \cong A \otimes \mathcal{Z} \Rightarrow \dim_{nuc} A < \infty$, assuming unique trace.
- Bosa–Brown–Sato–T–White–Winter '19: $A \cong A \otimes \mathcal{Z} \Rightarrow \dim_{nuc} A < \infty$, if $T(A)$ is a Bauer simplex.
- CETWW '21: $A \cong A \otimes \mathcal{Z} \Rightarrow \dim_{nuc} A < \infty$ (nonunital case: Castillejos–Evington '20).

# Applications

Classification again:

**Theorem**

Simple separable unital C*-algebras with finite nuclear dimension which satisfy the UCT are classified by K-theory and traces.

It turns out that $\mathcal{Z}$-stability often easier than $\dim_{nuc} < \infty$.
If $G$ is countable and $G \curvearrowright X$ is a free minimal action on a finite dim. metrizable compact space then $C(X) \rtimes G$ is $\mathcal{Z}$-stable (hence classifiable) if:

- Kerr–Szabó '20: $G$ has subexponential growth;
- Kerr-Naryshkin '21: $G$ is elementary amenable;
- Conley–Jackson–Kerr–Marks–Seward–Tucker-Drop '18: $G$ is amenable, for a generic action;
- **Question**: for all amenable groups?
- Niu '19: weakening $\dim(X) < \infty$ to $\mathrm{mdim}(G \curvearrowright X) = 0$, and $G = \mathbb{Z}^d$.

$D \cong D \otimes D$

$M_{2^\infty}$ $\mathcal{Z}$

$\mathcal{O}_2$ $\mathcal{O}_\infty$

$C$(Cantor set) X

Strongly self-absorbing:

$D \to D \otimes 1 \subseteq D \otimes D$ is a.u.e. to an isomorphism