Dimension reduction and Jiang-Su stability

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University of Münster

Workshop on C^*-algebras, dynamics, and classification

The Toms-Winter conjecture

Fact (Rørdam, '04, based on Villadsen)

There exist 2 non-isomorphic simple, separable, unital, nuclear C^*-algebras with the same K-theory and traces.

Conjecture

For a simple, separable, unital, nonelementary, nuclear C^*-algebra A in the UCT class, the following are equivalent:

(i) A is $\mathcal{Z}$-stable;
(ii) A has finite nuclear dimension;
(iii) A has strict comparison of positive elements;
(iv) A is an inductive limit of nice building blocks (2-NCCW complexes, direct sums of $M_n \otimes \mathcal{O}_m \otimes C(\mathbb{T})$).

Moreover, the algebras satisfying (i)-(iv) are classifiable.

Note: the conjecture holds for Villadsen's algebras.

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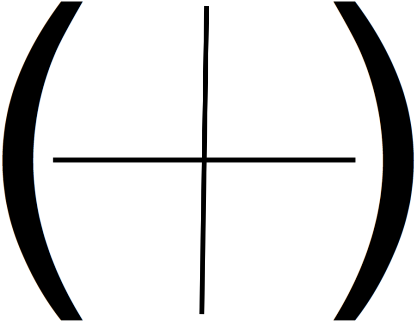

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M_{n^∞}-stable algebras (of the form $A \otimes M_{n^\infty}$) are very regular: UHF adds uniformity.

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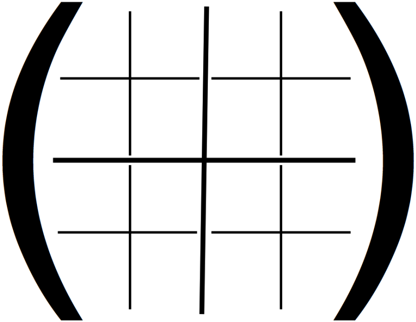

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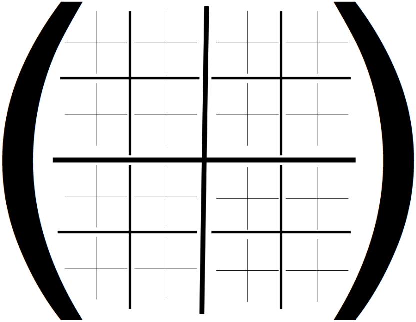

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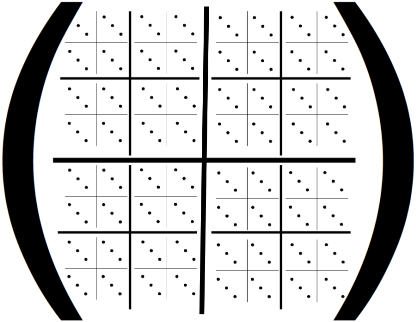

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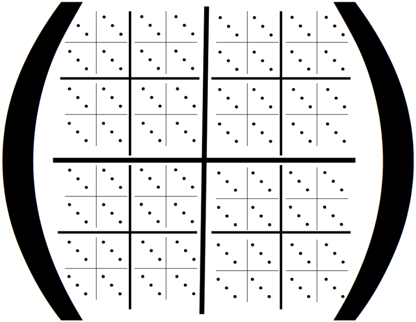

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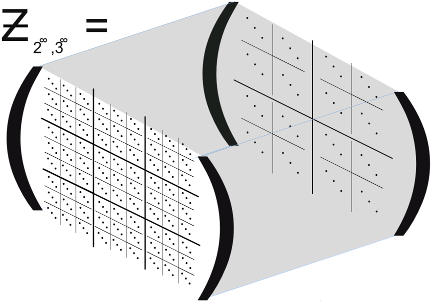

$\mathcal{Z}$ is a simple inductive limit of $\mathcal{Z}_{2^\infty,3^\infty}$, with unique trace.

Strongly self-absorbing; $\mathcal{Z}$-stability adds uniformity.

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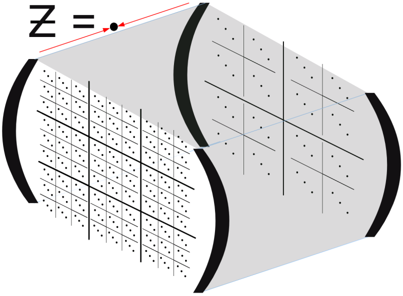

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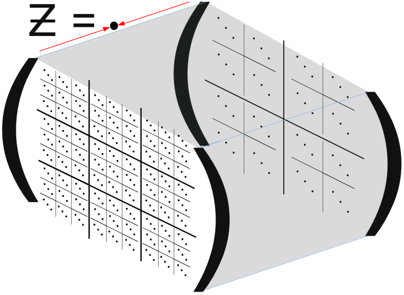

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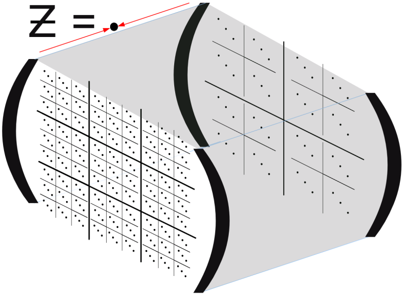

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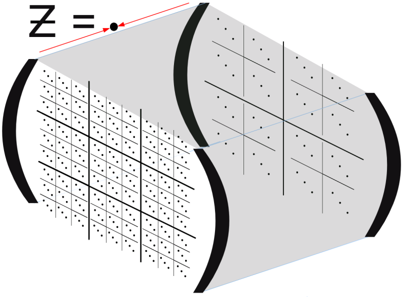

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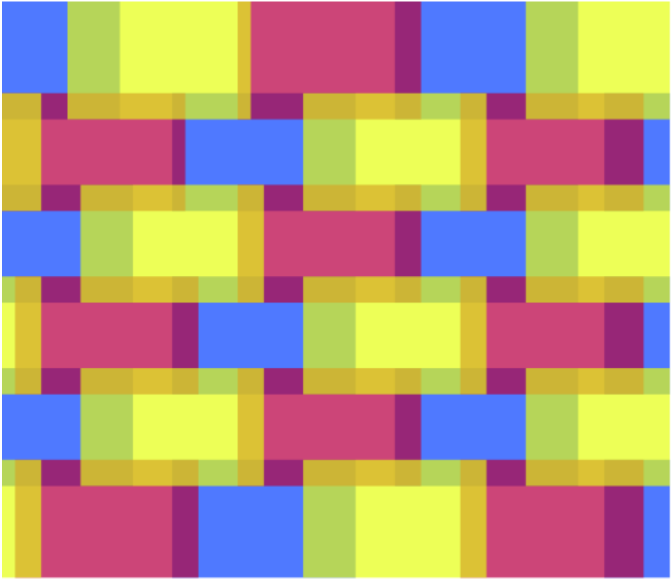

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Decomposition rank (Kirchberg-Winter '04)
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$$\begin{tikzcd}
A \arrow[rr, "="] \arrow[dr, "\text{c.p.c.}"'] & \arrow[d, phantom, "\text{approx. (pt-}\|\cdot\|\text{)}"] & A \\
& F \arrow[ur, "\text{c.p.c.}"'] &
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$ab = 0 \Rightarrow \phi(a)\phi(b) = 0$.

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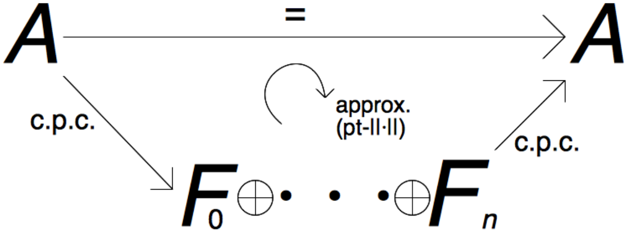

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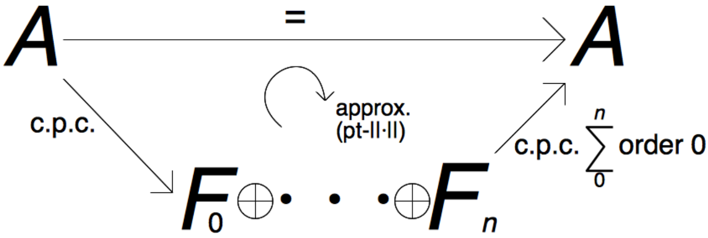

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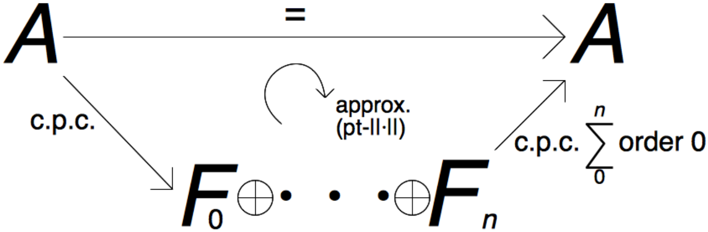

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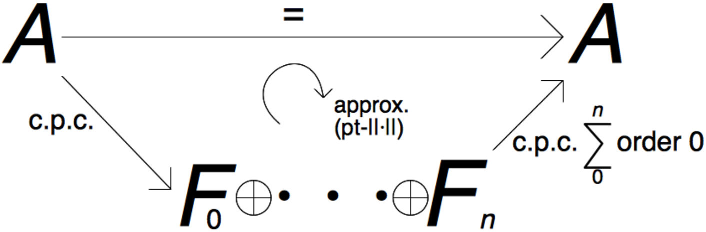

Nuclear dimension is defined by a slight tweaking of the definition of decomposition rank.

While $\mathrm{dr}\,(A) < \infty$ implies A is quasidiagonal, $\dim_{nuc}(\mathcal{O}_n) = 1$ (for $n < \infty$) for example.

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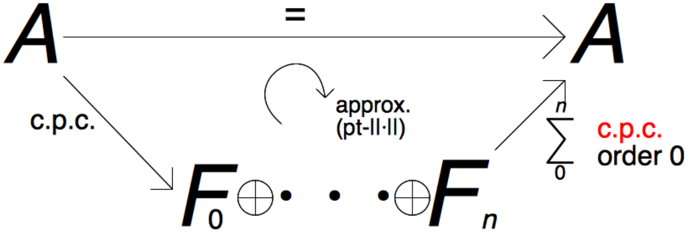

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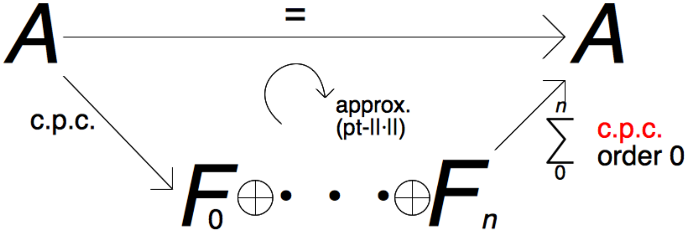

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$$\Big\Downarrow \text{ Rørdam ('04)}$$

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$\Downarrow$ $\Uparrow$ with additional hypotheses:
Winter ('12), Matui-Sato (arXiv '11),
Toms-White-Winter

strict comparison

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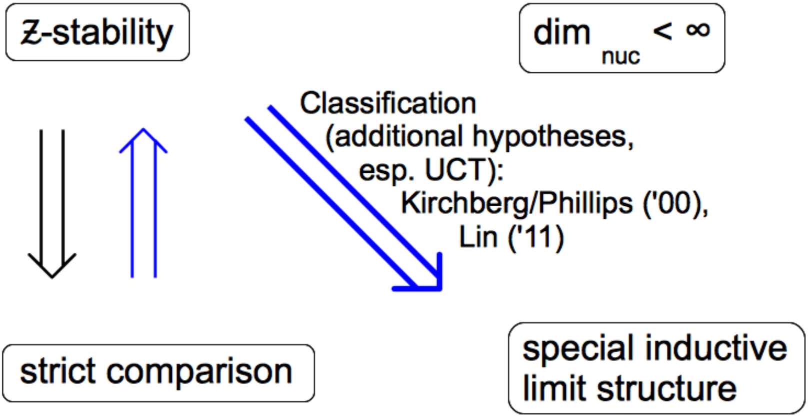

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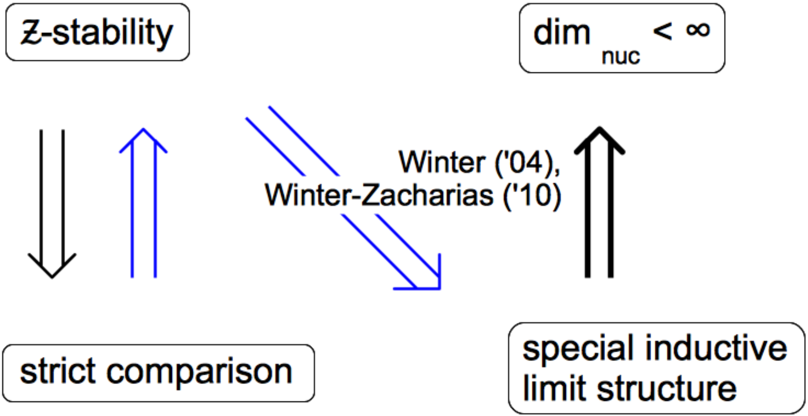

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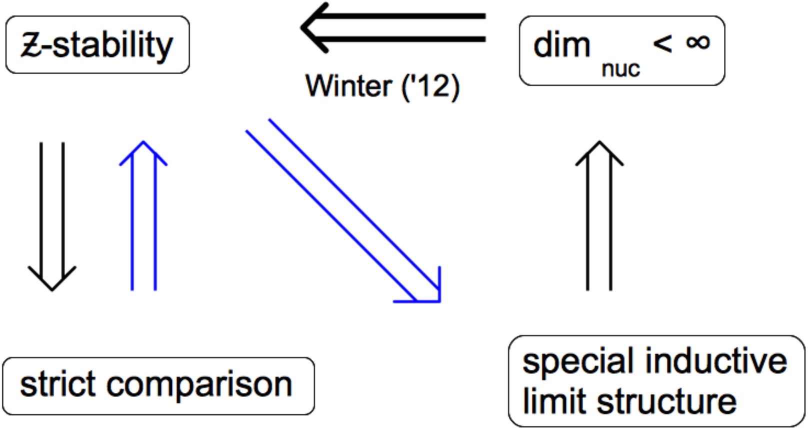

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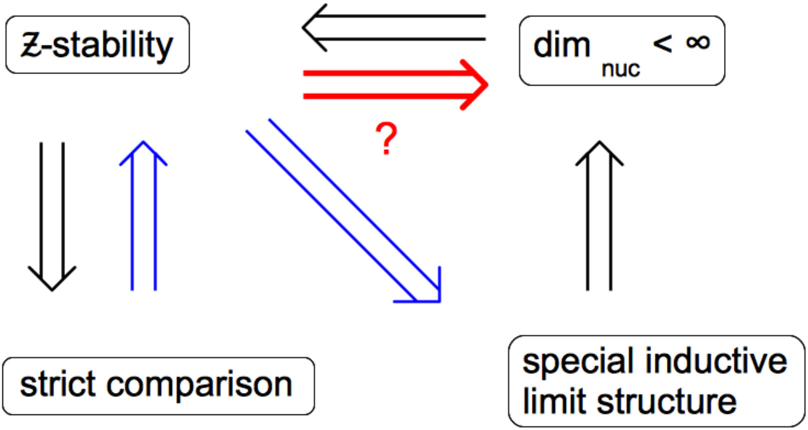

$\mathcal{Z}$-stable $\overset{?}{\Rightarrow}$ finite nuclear dimension is a question of dimension reduction, which has some history.

Toms' example (cf. Villadsen)

There exists a simple C^*-algebra A with infinite nuclear dimension, yet $\mathrm{dr}\,(A \otimes \mathcal{Z}) \leq 1$.

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A key point in the proof is establishing the following:

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$C_0(X, \mathbb{C} \cdot 1_{n^\infty}) \subset C_0(X, M_{n^\infty})$ can be approx. factorized as

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where ψ, ϕ are c.p.c. and ϕ is order zero when restricted to $C_0(Y, \mathcal{O}_2)$ or F.

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Edit added after the talk: The lemma may be false as stated for general (compact Hausdorff) X (the last line in the next slide isn't accurate). However, it is true for $X = [0,1]^d$, and the idea of local approximation does allow the theorem (with $\mathcal{Z}$ replaced by M_{n^∞}) to be proven using the lemma in this weakened form.

Proof (of lemma)

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Reduce to the case $X = [0,1]$:

If we have it for $X = [0,1]$, then we take products to get it for $X = [0,1]^d$.

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General X reduces to $[0,1]^d$ by local approximation.

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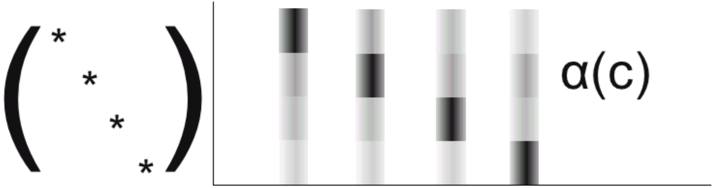

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Get orthogonal positive elements a_1, a_2 such that $a_1 + a_2 + \alpha(c) = 1$.

Repeat, $2 \to m$ so that each a_i has small support.

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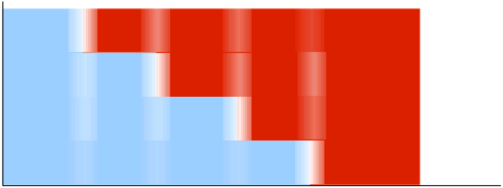

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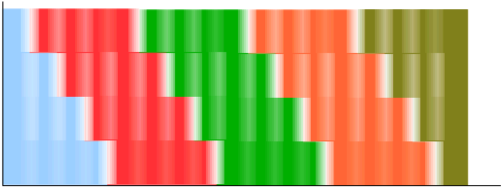

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Define the other map in a natural way, using c.

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Can we say more about the structure of $C(X) \subset C(X, \mathcal{Z})$?
Does it (approx.) factorize through $C(Y)$ with $\dim Y$ small?

Theorem (Santiago '12)

$C(X, \mathcal{W})$ is approximated by 1-NCCW complexes.

Question

Is $\dim_{nuc}(A \otimes \mathcal{Z}) < \infty$ for every nuclear C^*-algebra A?
Equivalently, is $\dim_{nuc}(A \otimes \mathcal{Z})$ universally bounded for such A?

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Can we approximate $C(X)$ inside $C(X, M_n)$ in a 2-dimensional way (3 colours)? At least, $< \dim X$ dimensions? Or is it necessary to put $C(X)$ into $C(X, M_{n^\infty})$?

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