$\mathcal{Z}$-stability, Property Γ, Partitions of Unity and Nuclear Dimension

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(joint work with Jorge Castillejos, Samuel Evington and Wilhelm Winter)

Following the complete classification of non-elementary, simple, separable and unital C*-algebras of finite nuclear dimension by the Elliott invariant, it is a major task to identify which simple nuclear C*-algebras have finite nuclear dimension.

Conjecture (Toms-Winter)**.** *Let A be a non-elementary, simple, separable, unital and nuclear* C**-algebra. The following are equivalent:*

(1) *A has finite nuclear dimension;*
(2) *A absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially;*
(3) *A has strict comparison of positive elements.*

A stronger form of the conjecture also predicts that when A is stably finite (1) can be replaced by the stronger condition:

($1'$) *A has finite decomposition rank.*

This conjecture has seen substantial work: the implications (1)$\Rightarrow$(2)$\Rightarrow$(3) hold in general (due to Winter and Rørdam respectively). The directions (3)$\Rightarrow$(2)$\Rightarrow$(1) would represent the full force of Connes' characterisations of injectivity. Partial results are known either assuming A has particular internal approximations, or under assumptions on the trace space of A.

Recent developments in the structure of crossed product C*-algebras, show the importance of being able to access classification from Jiang-Su stability. Despite the massive progress that has been made in obtaining finite nuclear dimension for crossed products through dynamical notions of dimension (such as the Rohklin dimension), it now seems likely that the class of groups for which this approach will succeed will be relatively limited. In contrast Kerr has identified a dynamical condition (almost finiteness), which gives $\mathcal{Z}$-stability for the crossed product (see his talk in this workshop), and, among things, with Conley, Jackson, Marks, Seward and Tucker-Drob, he has shown that this holds for generic free minimal actions of elementary amenable groups.

Murray and von Neumann's Property Γ

In their foundational work on II_1 factors, Murray and von Neumann introduced *property* Γ in order to distinguish the free group factor(s) from the hyperfinite II_1 factor. In today's language a separably acting II_1 factor $\mathcal{M}$ has property Γ if and only if the central sequence algebra $\mathcal{M}^\omega \cap \mathcal{M}'$ is non-trivial. In contrast, Akemann and Pedersen showed that the only separable C*-algebras with no non-trivial *norm* approximately central sequences are of continuous trace. In order to make a useful definition for simple C*-algebras, we use Diximer's characterisation: a II_1 factor $\mathcal{M}$ has property Γ if for each (or equivalently for some) $k \geq 2$, there are k pairwise orthogonal projections in $\mathcal{M}^\omega \cap \mathcal{M}'$ each of trace $1/k$. This formulation has been used by Christensen and Pisier to establishing the Kadison's similarity property in the presence of property Γ, and Ge and Popa use it to show these factors are singly generated.

In order to state our definition, recall that if A is a separable C*-algebra with ultrapower A_ω, then the *limit traces*, denoted $T_\omega(A_\omega)$, on A_ω are those traces[1] of the form $\tau((x_n)_{n=1}^\infty) = \lim_{n\to\omega} \tau_n(x_n)$, for some sequence $(\tau_n)_{n=1}^\infty$ of traces on A.

Definition. We say that a simple unital C*-algebra A has *property* Γ if and only if, for each $k \geq 2$, there exist pairwise orthogonal positive contractions $e_1, \dots, e_k$ in $A_\omega \cap A'$ such that

$$\tau(e_i a) = \frac{1}{k}\tau(a), \quad a \in A,\ \tau \in T_\omega(A_\omega), i = 1, \dots, k. \tag{1}$$

As with II_1 factors, it suffices to verify property Γ for some $k \geq 2$. Also, when $\partial T_e(A)$ is compact, it suffices to verify (1) with $a = 1_A$.

It is open whether all simple, separable, unital and nuclear C*-algebras have property Γ. For our purposes, there are two important classes of examples:

- All $\mathcal{Z}$-stable separable unital C*-algebras have property Γ.
- All separable unital nuclear C*-algebras with no finite dimensional representations whose tracial boundary is compact and of finite covering dimension have property Γ.

Our main result shows that property Γ is precisely the condition needed to establish the Toms-Winter conjecture.

Theorem. *The Toms-Winter conjecture holds under the additional assumption of property* Γ. *In particular (2)⇒(1) holds in general. Furthermore, when A is stably finite,* (1′) *is additionally equivalent to conditions* (1) *and* (2) *precisely when all traces on A are quasidiagonal.*

Partitions of unity

In our earlier work [1] with Brown, Bosa, Sato and Winter, we established (2)⇒(1) in the Toms-Winter conjecture under the hypothesis that the tracial boundary of A is compact. One major ingredient in this work, and the fundamental reason we required this compactness, is the ability to use $\mathcal{Z}$-stability to 'glue' together elements which have good tracial behaviour locally across the entire trace space. In [1] this was achieved using Ozawa's theory of W*-bundles, which relies on a compact tracial boundary. In this setting $\mathcal{Z}$-stability gives rise to a trivial bundle, and then the required gluing can be performed using a partition of unity argument. Abstracting a version of this argument leads to the following definition.

Definition. Let A be a separable C*-algebra. Say that A has *complemented partitions of unity* if for any finite family $a_1, \dots, a_n$ of positive contractions in A, and $\delta > 0$ such that

$$\sup_{\tau \in T(A)} \min_{i=1,\dots,n} \tau(a_i) < \delta,$$

[1] For us, at least here, traces are always states.

there exist pairwise orthogonal positive contractions $e_1, \ldots, e_n \in A_\omega \cap A'$ such that:

(1) $\tau(\sum_{i=1}^n e_i) = 1$ for all $\tau \in T_\omega(A_\omega)$;
(2) $\tau(a_i e_i) \leq \delta\tau(e_i)$ for all $\tau \in T_\omega(A_\omega)$.

The first property says that the e_i form a tracial partition of unity; the second is that, in a suitable sense, they complement the a_i (thought of as being "subordinate to the complement of the support of $(a_i - \delta)_+$," as might be the case for a classical partition of unity on $C(X)$). In applications the orthogonality and asymptotic centrality of the e_i are vital. For example, suppose A is nuclear and has no finite dimensional representations. Then one can use compactness of $T(A)$ to obtain finitely many order zero maps $\phi_1, \ldots, \phi_n : M_2 \to A$ such that for each $\tau \in T(A)$, there is some i with $\tau(\phi_i(1))$ is large. Taking $a_i = 1_A - \phi_i(1)$, we can use a complemented partition of unity (e_i) to define an order zero map $\phi(\cdot) = \sum_{i=1}^n e_i^{1/2}\phi_i(\cdot)e_i^{1/2}$, so that $\tau(\phi(1))$ is globally large. The pairwise orthogonality of the e_i and asymptotic centrality ensure that this really is an order zero map.

Our main technical result is that property Γ gives rise to these complemented partitions of unity, at least when A is nuclear.[2] The main theorem is then obtained using the complemented partitions of unity to replace the gluing arguments of [1].

Obtaining partitions of unity

We end this note with a brief description of our main technical result:

Theorem. *Let A be a separable nuclear* C^**-algebra with property* Γ*. Then A has complemented partitions of unity.*

We fix $a_1, \ldots, a_n$ and δ as in the definition of complemented partitions of unity. The first stage is to use a refined version of the completely positive approximation property, due to Brown, Carrión and SW. We can then push a partition of unity obtained at the level of the approximation back to A in a fashion compatible with the multiplication. This produces e_i as in the definition of complemented partitions of unity, except for the fact that the e_i need not be pairwise orthogonal.

The second step is to use property Γ to convert the e_i in the first step into tracial projections, and then place these underneath pairwise orthogonal approximately central elements (as in the definition of property Γ). This ensures that the resulting e_i are pairwise orthogonal, but it comes at a cost: they only have $\tau(\sum_{i=1}^n e_i) = \frac{1}{n}$ for all $\tau \in T_\omega(A_\omega)$. The argument is then repeated underneath the tracial projection $1 - \sum_{i=1}^n e_i$, obtaining another $\frac{1}{n}$ of the remaining trace. Carrying on in this way gives the required partition of unity: the point being that the geometric series $\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{1}{n} + \left(1 - \left(1 - \frac{1}{n}\right)\frac{1}{n}\right)\frac{1}{n} + \cdots = 1$.

References

[1] N. Brown, J. Bosa, Y. Sato, A. Tikuisis, S. White and W. Winter, *Covering dimension of* C^**-algebras and 2-coloured classification*, Mem. Amer. Math. Soc., to appear. arXiv:1506.03974.

[2] When the tracial boundary of A is compact, we do not need a nuclearity hypothesis.